

## MATH 220: PDES AND BOUNDARIES

We have used the Fourier transform and other tools (factoring the PDE) to solve PDEs on  $\mathbb{R}^n$ . We now study how we can use these results to solve problems on the half space, or indeed on intervals, cubes, etc.

As you have shown on your problem set, the solution of the wave equation on  $\mathbb{R}^n$  (so  $u$  is a function on  $\mathbb{R}_x^n \times \mathbb{R}_t$ ) is even, resp. odd, in  $x_n$  if the initial conditions and the inhomogeneity are even, resp. odd in  $x_n$ . That is, write  $x = (x', x_n)$  where  $x' = (x_1, \dots, x_{n-1})$ . The wave equation is

$$u_{tt} - c^2 \Delta u = f, \quad u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x).$$

If

$$f(x', x_n, t) = \pm f(x', -x_n, t), \quad \phi(x', x_n) = \pm \phi(x', -x_n), \quad \psi(x', x_n) = \pm \psi(x', -x_n)$$

for all  $x$  and  $t$ , i.e. if  $f, \phi, \psi$  are all even (+), resp. odd (-), functions of  $x_n$ , then  $u$  is an even, resp. odd function of  $x_n$  as well, i.e.

$$u(x', x_n, t) = \pm u(x', -x_n, t).$$

Recall that this was based on considering  $u(x', x_n, t) \mp u(x', -x_n, t)$ , and showing that it solves the homogeneous wave equation with 0 initial conditions. You also showed that if  $u$  is continuous, and is an odd function of  $x_n$ , then  $u(x', 0, t) = 0$  for all  $x'$  and  $t$ , while if  $u$  is a  $C^1$  and is an even function of  $x_n$ , show that  $\partial_{x_n} u(x', 0, t) = 0$  for all  $x'$  and  $t$ .

These observations reduce the solution of the wave equation in  $x_n > 0$  with either Dirichlet or Neumann boundary condition to solving the PDE on all of  $\mathbb{R}^n$ . For the sake of definiteness, suppose we want to solve the Dirichlet problem:

$$\begin{aligned} u_{tt} - c^2 \Delta u &= f, \quad x_n \geq 0, \\ u(x', 0, t) &= 0 \text{ (DBC)}, \\ u(x, 0) &= \phi(x), \quad u_t(x, 0) = \psi(x), \quad x_n \geq 0 \text{ (IC)}. \end{aligned}$$

Here  $f, \phi$  and  $\psi$  are given functions, defined in  $x_n \geq 0$  only. To solve the PDE, we consider the *odd extensions* of  $f, \phi, \psi$  in  $x_n$ , i.e. *define*

$$f_{\text{odd}}(x', x_n, t) = \begin{cases} f(x', x_n, t), & x_n \geq 0, \\ -f(x', -x_n, t), & x_n < 0, \end{cases},$$

and analogously

$$\phi_{\text{odd}}(x', x_n) = \begin{cases} \phi(x', x_n), & x_n \geq 0, \\ -\phi(x', -x_n), & x_n < 0, \end{cases},$$

with a similar definition for  $\psi$ . The resulting function is odd and continuous, provided that  $f(x', 0, t) = 0 = \phi(x', 0) = \psi(x', 0)$ , i.e. if the data are compatible with the boundary condition. Indeed, for  $x_n < 0$  then

$$\phi_{\text{odd}}(x', x_n) = -\phi(x', -x_n) = -\phi_{\text{odd}}(x', -x_n),$$

and similarly in all other cases.

Now let  $v$  be the solution of the wave equation on  $\mathbb{R}^n$  with these odd data:

$$\begin{aligned} v_{tt} - c^2 \Delta v &= f_{\text{odd}}, \\ v(x, 0) &= \phi_{\text{odd}}(x), \quad v_t(x, 0) = \psi_{\text{odd}}(x) \quad (\text{IC}). \end{aligned}$$

As we have seen,  $v$  is an odd function of  $x_n$ , hence  $v(x', 0, t) = 0$  for all  $x'$  and  $t$ . Now simply let  $u$  be the restriction of  $v$  to  $x_n \geq 0$ , so

$$u(x', x_n, t) = v(x', x_n, t), \quad x_n \geq 0.$$

Then  $u$  solves the PDE, satisfies the initial conditions as well as the Dirichlet boundary conditions, so we have solved our problem!

Concretely, if  $n = 1$ , we obtain the following result for the solution of the homogeneous wave equation with Dirichlet BC:

$$v(x, t) = \frac{1}{2}(\phi_{\text{odd}}(x + ct) + \phi_{\text{odd}}(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{\text{odd}}(\sigma) d\sigma.$$

Then  $u$  is the restriction of  $v$  to  $x \geq 0$ , so it remains to work out these formulae in terms of  $\phi$  and  $\psi$  themselves. If  $x \geq 0$ ,  $t \geq 0$  and  $x \geq ct$  then  $x - ct \geq 0$ , so we simply have

$$u(x, t) = \frac{1}{2}(\phi(x + ct) + \phi(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(\sigma) d\sigma, \quad x \geq ct \geq 0,$$

i.e. the standard solution formula, which was to be expected in view of the propagation speed of waves: if  $x > ct$ , the effects of the boundary cannot be felt yet. On the other hand, if  $x \geq 0$ ,  $t \geq 0$  and  $x < ct$  then  $x - ct < 0$  (but  $x + ct \geq 0$  still!), so

$$\begin{aligned} u(x, t) &= \frac{1}{2}(\phi(x + ct) - \phi(ct - x)) - \frac{1}{2c} \int_{x-ct}^0 \psi(-\sigma) d\sigma + \frac{1}{2c} \int_0^{x+ct} \psi(\sigma) d\sigma \\ &= \frac{1}{2}(\phi(x + ct) - \phi(ct - x)) + \frac{1}{2c} \int_{ct-x}^0 \psi(\sigma) d\sigma + \frac{1}{2c} \int_0^{x+ct} \psi(\sigma) d\sigma \\ &= \frac{1}{2}(\phi(x + ct) - \phi(ct - x)) + \frac{1}{2c} \int_{ct-x}^{x+ct} \psi(\sigma) d\sigma. \end{aligned}$$

We can solve the Neumann problem similarly:

$$\begin{aligned} u_{tt} - c^2 \Delta u &= f, \quad x_n \geq 0, \\ (\partial_{x_n} u)(x', 0, t) &= 0 \quad (\text{NBC}), \\ u(x, 0) &= \phi(x), \quad u_t(x, 0) = \psi(x), \quad x_n \geq 0 \quad (\text{IC}). \end{aligned}$$

Again,  $f$ ,  $\phi$  and  $\psi$  are given functions, defined in  $x_n \geq 0$  only. To solve the PDE, we consider the *even extensions* of  $f, \phi, \psi$  in  $x_n$ , i.e. *define*

$$f_{\text{even}}(x', x_n, t) = \begin{cases} f(x', x_n, t), & x_n \geq 0, \\ f(x', -x_n, t), & x_n < 0, \end{cases},$$

and analogously

$$\phi_{\text{even}}(x', x_n) = \begin{cases} \phi(x', x_n), & x_n \geq 0, \\ \phi(x', -x_n), & x_n < 0, \end{cases},$$

with a similar definition for  $\psi$ . The resulting function is even and  $C^1$ , provided that  $f_{x_n}(x', 0, t) = 0 = \phi_{x_n}(x', 0) = \psi_{x_n}(x', 0)$ , which again means that the data

are compatible with the boundary condition. We again check, e.g. for  $f$  this time, that for  $x_n < 0$

$$f_{\text{even}}(x', x_n, t) = f(x', -x_n, t) = f_{\text{even}}(x', -x_n, t),$$

and similarly in all other cases.

We now let  $v$  be the solution of the wave equation on  $\mathbb{R}^n$  with these even data:

$$\begin{aligned} v_{tt} - c^2 \Delta v &= f_{\text{even}}, \\ v(x, 0) &= \phi_{\text{even}}(x), \quad v_t(x, 0) = \psi_{\text{even}}(x) \quad (\text{IC}). \end{aligned}$$

Now  $v$  is an even function of  $x_n$ , hence  $v_{x_n}(x', 0, t) = 0$  for all  $x'$  and  $t$ . We finally let  $u$  be the restriction of  $v$  to  $x_n \geq 0$ , so

$$u(x', x_n, t) = v(x', x_n, t), \quad x_n \geq 0.$$

Then  $u$  solves the PDE, satisfies the initial conditions as well as the Neumann boundary conditions, so we have solved our problem!

If  $n = 1$ , a calculation analogous to the DBC gives for the homogeneous PDE in  $x - ct < 0$ ,  $x \geq 0$ ,  $t \geq 0$  (the usual formula still holds if  $x - ct \geq 0$ )

$$\begin{aligned} u(x, t) &= \frac{1}{2}(\phi(x + ct) + \phi(ct - x)) + \frac{1}{2c} \int_{x-ct}^0 \psi(-\sigma) d\sigma + \frac{1}{2c} \int_0^{x+ct} \psi(\sigma) d\sigma \\ &= \frac{1}{2}(\phi(x + ct) + \phi(ct - x)) + \frac{1}{c} \int_0^{ct-x} \psi(\sigma) d\sigma + \frac{1}{2c} \int_{ct-x}^{x+ct} \psi(\sigma) d\sigma. \end{aligned}$$

The main reason our method worked is that for any function (or indeed distribution)  $u$ , letting  $u_-(x', x_n, t) = u(x', -x_n, t)$ , we have

$$((\partial_t^2 - c^2 \Delta)u_-)(x', x_n, t) = ((\partial_t^2 - c^2 \Delta)u)(x', -x_n, t).$$

Since

$$(\partial_x^\alpha \partial_t^k u_-)(x', x_n, t) = (\partial_x^\alpha \partial_t^k u)(x', -x_n, t)$$

for all  $\alpha \in \mathbb{N}^{n-1}$ ,  $k \in \mathbb{N}$ , as these differentiations are not in the  $x_n$  variable, this boils down to

$$(\partial_{x_n}^k u_-)(x', x_n, t) = (-1)^k (\partial_{x_n}^k u)(x', -x_n, t),$$

and that only even number (in this case  $k = 2$ )  $x_n$ -derivatives of  $u$  enter into the wave operator. Thus, for instance, the wave equation for the bi-Laplacian,  $(\partial_t^2 + c^2 \Delta^2)u = 0$  with appropriate boundary conditions, namely either the function and its second derivative vanishing, in which case we could use odd extension, or the first and third derivatives vanishing, in which case we could use even extensions, could also be treated by this method.

The heat equation is completely analogous. Thus, to solve, for instance,

$$\begin{aligned} u_t - k \Delta u &= f, \quad x_n \geq 0, \\ u(x', 0, t) &= 0 \quad (\text{DBC}), \\ u(x, 0) &= \phi(x), \quad x_n \geq 0 \quad (\text{IC}), \end{aligned}$$

where  $f$  and  $\phi$  are given functions, defined in  $x_n \geq 0$  only, we consider the *odd extensions*  $f_{\text{odd}}$  and  $\phi_{\text{odd}}$  of  $f$  and  $\phi$  in  $x_n$ , and solve

$$\begin{aligned} v_t - k \Delta v &= f_{\text{odd}}, \\ v(x, 0) &= \phi_{\text{odd}}(x) \quad (\text{IC}). \end{aligned}$$

Once we check that  $v$  is an odd function of  $x_n$ , hence  $v(x', 0, t) = 0$  for all  $x'$  and  $t$ , we are done as before: letting  $u$  be the restriction of  $v$  to  $x_n \geq 0$ , so

$$u(x', x_n, t) = v(x', x_n, t), \quad x_n \geq 0,$$

$u$  solves the PDE, satisfies the initial conditions as well as the Dirichlet boundary conditions.

It remains to check that  $v$  is indeed an odd function of  $x_n$ . This can be done as for the wave equation. Namely, consider  $w(x', x_n, t) = v(x', x_n, t) + v(x', -x_n, t)$ . Since  $v$  solves the inhomogeneous heat equation with odd  $f$ ,  $w$  solves the homogeneous heat equation:

$$\begin{aligned} & (w_t - k\Delta w)(x, t) \\ &= v_t(x', x_n, t) + v_t(x', -x_n, t) - k(\Delta v)(x', x_n, t) - k(\Delta v)(x', -x_n, t) \\ &= f_{\text{odd}}(x', x_n, t) + f_{\text{odd}}(x', -x_n, t) = 0, \end{aligned}$$

since  $f_{\text{odd}}$  is odd. In addition,  $w$  has vanishing initial condition:

$$w(x, 0) = v(x', x_n, 0) + v(x', -x_n, 0) = \phi_{\text{odd}}(x', x_n) + \phi_{\text{odd}}(x', -x_n) = 0$$

as  $\phi_{\text{odd}}$  is odd. Assuming, for instance, that  $v(x, t) \rightarrow 0$  as  $|x| \rightarrow \infty$ , hence the same holds for  $w$  the maximum principle, as on Problem Set 4, shows that  $w$  vanishes, so  $v$  is indeed odd.

We would still need to check the needed decay of  $v$  (assuming decay on  $f$  and  $\phi$ ), but this follows from our solution formula. Indeed, we can see *directly* from the solution formula, at this point for  $f = 0$ , that the  $v$  we constructed is odd. Indeed, the solution formula is

$$v(x, t) = (4\pi kt)^{-n/2} \int_{\mathbb{R}^n} e^{-|x-y|^2/(4kt)} \phi_{\text{odd}}(y) dy,$$

so

$$\begin{aligned} v(x', -x_n, t) &= (4\pi kt)^{-n/2} \int_{\mathbb{R}^n} e^{-|x'-y|^2/(4kt)} e^{-(x_n-y_n)^2/(4kt)} \phi_{\text{odd}}(y', y_n) dy \\ &= (4\pi kt)^{-n/2} \int_{\mathbb{R}^n} e^{-|x'-y|^2/(4kt)} e^{-(-x_n+y_n)^2/(4kt)} \phi_{\text{odd}}(y', -y_n) dy \\ &= -(4\pi kt)^{-n/2} \int_{\mathbb{R}^n} e^{-|x'-y|^2/(4kt)} e^{-(x_n-y_n)^2/(4kt)} \phi_{\text{odd}}(y', y_n) dy \\ &= -v(x', x_n, t), \end{aligned}$$

where the second equality is a change of variables from  $y_n$  to  $-y_n$ , and the third equality uses that  $\phi_{\text{odd}}$  is odd.

We obtain an explicit solution formula this way (for  $x \geq 0$ ,  $t \geq 0$ ):

$$\begin{aligned}
u(x, t) &= -(4\pi kt)^{-n/2} \int_{y_n \leq 0} e^{-|x-y|^2/(4kt)} \phi(y', -y_n) dy \\
&\quad + (4\pi kt)^{-n/2} \int_{y_n \geq 0} e^{-|x-y|^2/(4kt)} \phi(y) dy, \\
&= -(4\pi kt)^{-n/2} \int_{y_n \geq 0} e^{-|x'-y'|^2/(4kt)} e^{-(x_n+y_n)^2/(4kt)} \phi(y) dy \\
&\quad + (4\pi kt)^{-n/2} \int_{y_n \geq 0} e^{-|x-y|^2/(4kt)} \phi(y) dy, \\
&= (4\pi kt)^{-n/2} \\
&\quad \int_{y_n \geq 0} e^{-|x'-y'|^2/(4kt)} \left( e^{-(x_n-y_n)^2/(4kt)} - e^{-(x_n+y_n)^2/(4kt)} \right) \phi(y) dy \\
&= \int_{y_n \geq 0} G(x, y, t) \phi(y) dy,
\end{aligned}$$

where

$$G(x, y, t) = (4\pi kt)^{-n/2} e^{-|x'-y'|^2/(4kt)} \left( e^{-(x_n-y_n)^2/(4kt)} - e^{-(x_n+y_n)^2/(4kt)} \right),$$

and where the second equality is a change of variables from  $y_n$  to  $-y_n$ . Note that this is still an integral over the space,  $y_n \geq 0$  in this case, but the Gaussian has been amended by subtracting its ‘reflection’ around the  $x_n = 0$  plane,  $e^{-|x'-y'|^2/(4kt)} e^{-(x_n+y_n)^2/(4kt)}$ . It is also instructive to notice that we can see that the boundary condition is indeed satisfied directly from

$$G(x', 0, y, t) = (4\pi kt)^{-n/2} e^{-|x'-y'|^2/(4kt)} \left( e^{-(-y_n)^2/(4kt)} - e^{-y_n^2/(4kt)} \right) = 0$$

for all  $x', y, t$ .

Laplace’s equation works similarly. Thus, to solve

$$\begin{aligned}
\Delta u &= f, \quad x_n \geq 0, \\
u(x', 0) &= 0 \quad (\text{DBC}),
\end{aligned}$$

where  $f$  is defined for  $x_n \geq 0$  only, we take the odd extension  $f_{\text{odd}}$  of  $f$  to  $\mathbb{R}^n$ , solve

$$\Delta v = f_{\text{odd}}$$

on  $\mathbb{R}^n$ , and let  $u$  be the restriction of  $v$  to  $x_n \geq 0$ . If  $f$  decays at infinity, so does  $f_{\text{odd}}$ , and by the maximum principle,  $\Delta v = f_{\text{odd}}$  has a unique solution that decays at infinity. Explicitly, as you show on your problem set, one has

$$v(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^n} |x-y|^{-1} f_{\text{odd}}(y) dy,$$

so, letting  $x_- = (x', -x_n)$  be the reflection of  $x$  around the  $x_n$ -plane, one has

$$\begin{aligned}
u(x', x_n) &= -\frac{1}{4\pi} \int_{y_n \geq 0} |x - y|^{-1} f(y) dy + \frac{1}{4\pi} \int_{y_n \leq 0} |x - y|^{-1} f(y', -y_n) dy \\
&= -\frac{1}{4\pi} \int_{y_n \geq 0} |x - y|^{-1} f(y) dy \\
&\quad + \frac{1}{4\pi} \int_{y_n \geq 0} (|x' - y'|^2 + (x_n + y_n)^2)^{-1/2} f(y', y_n) dy \\
&= -\frac{1}{4\pi} \int_{y_n \geq 0} |x - y|^{-1} f(y) dy \\
&\quad + \frac{1}{4\pi} \int_{y_n \geq 0} (|x' - y'|^2 + (-x_n - y_n)^2)^{-1/2} f(y', y_n) dy \\
&= -\frac{1}{4\pi} \int_{y_n \geq 0} (|x - y|^{-1} - |x_- - y|^{-1}) f(y) dy \\
&= \int_{y_n \geq 0} G(x, y) f(y) dy,
\end{aligned}$$

where

$$G(x, y) = -\frac{1}{4\pi} (|x - y|^{-1} - |x_- - y|^{-1}),$$

and where the second equality is a change of variables from  $y_n$  to  $-y_n$ , while the third is rewriting

$$(|x' - y'|^2 + (x_n + y_n)^2)^{-1/2} = (|x' - y'|^2 + (-x_n - y_n)^2)^{-1/2} = |x_- - y|^{-1}.$$

Again, note that for all  $x'$  and all  $y$ ,  $G(x', 0, y) = 0$  since  $x_- = x$  if  $x_n = 0$ , so indeed the boundary condition is satisfied. In physics,  $\Delta u = f$  finds the electrostatic potential  $u$  associated to a charge distribution  $f$ . Correspondingly, considering  $f_{\text{odd}}$  means that we place imaginary charges in  $x_n < 0$ , with opposite signs to those in  $x_n > 0$  and solve the problem in the whole space. Thus, this method is also called the method of images.

These results can easily be generalized to quadrants, etc. For instance, to solve the heat equation in the quadrant  $x_{n-1}, x_n \geq 0$ , write  $x = (x', x_{n-1}, x_n)$ ,  $x' = (x_1, \dots, x_{n-2})$ . Impose, for instance, Dirichlet boundary condition at  $x_n = 0$ , Neumann at  $x_{n-1} = 0$ , so the PDE is

$$\begin{aligned}
u_t - k\Delta u &= f, \quad x_{n-1}, x_n \geq 0, \\
u(x', x_{n-1}, 0, t) &= 0 \text{ (DBC)}, \quad u(x', 0, x_n, t) = 0 \text{ (NBC)} \\
u(x, 0) &= \phi(x), \quad x_{n-1}, x_n \geq 0 \text{ (IC)},
\end{aligned}$$

where  $f$  and  $\phi$  are defined if both  $x_{n-1}$  and  $x_n$  are  $\geq 0$ . The method is then to extend  $\phi$  and  $f$  to all of  $\mathbb{R}_x^n$  in such a manner that the extensions  $\phi_{\text{ext}}$ , resp.  $f_{\text{ext}}$ , are odd in  $x_n$  and even in  $x_{n-1}$ . This can be achieved by first extending  $\phi$  (and similarly  $f$ ) to be even in  $x_{n-1}$ , defined now for only  $x_n \geq 0$ :

$$\phi_{\text{even}}(x', x_{n-1}, x_n) = \begin{cases} \phi(x', x_{n-1}, x_n), & x_{n-1} \geq 0, \\ \phi(x', -x_{n-1}, x_n), & x_{n-1} < 0, \end{cases},$$

and then extending  $\phi_{\text{even}}$  to be defined on all of  $\mathbb{R}^n$  by making it odd in  $x_n$ :

$$\phi_{\text{ext}}(x', x_{n-1}, x_n) = \begin{cases} \phi_{\text{even}}(x', x_{n-1}, x_n), & x_n \geq 0, \\ -\phi_{\text{even}}(x', x_{n-1}, -x_n), & x_n < 0, \end{cases}.$$

Of course, we could have done the extensions in the opposite order. Then the extensions  $\phi_{\text{ext}}$  and  $f_{\text{ext}}$  are odd in  $x_n$  and even in  $x_{n-1}$ , as desired.

We again let  $v$  be the solution of the PDE with the extended data on all of  $\mathbb{R}^n$ :

$$\begin{aligned} v_t - k\Delta v &= f_{\text{ext}}, \\ v(x, 0) &= \phi_{\text{ext}}(x) \text{ (IC)}. \end{aligned}$$

It is now even in  $x_{n-1}$ , odd in  $x_n$ , so letting  $u(x, t) = v(x, t)$  if  $x_{n-1}, x_n \geq 0$ , we have solved the PDE.

When solving PDEs in half spaces, quadrants, etc., it is sometimes easier to read off properties of the solution of the PDE directly from the solution  $v$  on  $\mathbb{R}^n$ , rather than trying to work out an explicit formula in the region. For instance, the solution of the homogeneous heat equation on  $\mathbb{R}^n \times (0, \infty)$  is  $C^\infty$  in  $t > 0$ , even with merely bounded continuous, or  $L^1$ , or tempered distributional, initial data, hence the same remains true for the heat equation in half space with either Dirichlet or Neumann initial data.

On the other hand, for the wave equation, singularities propagate along characteristics. If  $n = 1$ , with either Dirichlet or Neumann boundary conditions, if the original initial condition is  $C^\infty$  at some  $x$  with  $x > 0$ , the extension will be  $C^\infty$  at  $x$  and  $-x$ , and conversely, if the original initial condition is not  $C^\infty$  at some  $x$  with  $x > 0$ , the extension will not be  $C^\infty$  at  $x$  or  $-x$ . The situation at  $x = 0$  is more delicate; e.g. the function  $\phi(x) = x^4$ ,  $x \geq 0$ , is a perfectly nice  $C^\infty$  function, but its odd extension is not:  $\phi_{\text{odd}}(x) = -x^4$  for  $x < 0$ , so  $\lim_{x \rightarrow 0^+} \phi^{(4)}(x) = 24$ , while  $\lim_{x \rightarrow 0^-} \phi^{(4)}(x) = -24$ , so its 4th derivative is not continuous. This may be avoided e.g. by assuming that all derivatives of  $\phi$  vanish at 0, or even more strongly,  $\phi$  vanishes identically near 0. Under either of these assumptions, the singularities of the extensions of  $\phi$  lie exactly at  $x$  and  $-x$  as  $x$  runs through the points on the positive half-line at which  $\phi$  is singular. Correspondingly, the solution of the wave equation on  $\mathbb{R}_x \times \mathbb{R}_t$  will be singular on the characteristics through the two points  $(x, 0)$  and  $(-x, 0)$  for each such  $x$ . This gives that singularities of solutions *reflect* from the boundary with the usual (equal angle) law of reflection: fixing  $x_0 > 0$  at which  $\phi$  is singular, in  $t > 0$  the characteristic through  $(x_0, 0)$  along which  $x$  is decreasing,  $x = x_0 - ct$  (or  $x + ct = x_0$ ) hits  $x = 0$  at  $t = x_0/c$ , which is the same place where the characteristic from  $(-x_0, 0)$  along which  $x$  is increasing,  $x = -x_0 + ct$ , hits  $x = 0$ . Thus, restricted to  $x \geq 0$ , the singularity seems to reflect from the boundary, since we do not ‘see’ the images that arose by our even or odd extensions.

While we conveniently had all our boundaries on a coordinate plane, this is by no means necessary. For instance, consider the PDE

$$\begin{aligned} u_t &= k\Delta_x u, \quad x_n \geq L, \\ u(x', 0, t) &= 0, \\ u(x, 0) &= \phi(x), \quad x_n \geq L. \end{aligned}$$

Introducing the new variable,  $y = (y', y_n)$  with  $y_n = x_n - L$  and  $y' = x'$ , for the function  $\tilde{u}(y', y_n, t) = u(y', y_n + L, t)$ , the PDE becomes

$$\begin{aligned} \tilde{u}_t &= k\Delta_y \tilde{u}, \quad y_n \geq 0, \\ \tilde{u}(y', 0, t) &= 0, \\ \tilde{u}(y', y_n, 0) &= \phi(y', y_n + L), \quad y_n \geq 0, \end{aligned}$$

which we can solve as above. Then we let  $u(x', x_n, t) = \tilde{u}(x', x_n - L, t)$  to get back the solution of the original equation.

Rather than going through this process, we could have obtained the the same result by extending  $u$  *directly* to an *odd function about the plane*  $x_n = L$ . The reflection of a point  $(x', x_n)$  about this plane is  $(x', L - (x_n - L)) = (x', 2L - x_n)$ , so the extension we would have considered is

$$\phi_{\text{ext}}(x', x_n) = \begin{cases} \phi(x', x_n), & x_n \geq L, \\ -\phi(x', 2L - x_n), & x_n < L, \end{cases}.$$

We can then solve the PDE on  $\mathbb{R}^n$  using these extended initial data, and restrict to  $x_n \geq L$  to get the desired solution.

We can also work on intervals, cubes, etc., using further iterated versions of these methods. As an example, consider the heat equation on  $[0, \ell]_x \times (0, \infty)_t$  with Dirichlet boundary condition. We have seen that we should take the odd extension around both  $x = 0$  and  $x = \ell$ . This can be simplified as follows: we take the odd  $2\ell$ -periodic extension of  $\phi$ . That is, first we extend  $\phi$ , which at first is defined on  $[0, \ell]$ , to an odd function on  $[-\ell, \ell]$ :

$$\phi_{\text{odd}}(x) = \begin{cases} \phi(x), & x \geq 0, \\ -\phi(-x), & x < 0. \end{cases}$$

Next, we extend  $\phi_{\text{odd}}$  to a  $2\ell$ -periodic function on  $\mathbb{R}$ , i.e. to a function  $\phi_{\text{ext}}$ , i.e. to a function satisfying

$$\phi_{\text{ext}}(x + 2\ell) = \phi_{\text{ext}}(x), \quad x \in \mathbb{R}.$$

This can be done, since any  $x \in \mathbb{R}$  can be translated by an integer multiple of  $2\ell$  so that the result lies in  $[-\ell, \ell]$ , and the result is a unique point except if we started at an odd multiple of  $\ell$  (in which case both  $\pm\ell$  are acceptable results). Moreover,  $\phi_{\text{ext}}$  is still odd, for if  $x \in \mathbb{R}$  and  $x - 2k\ell \in [-\ell, \ell]$  then  $-x + 2k\ell = -(x - 2k\ell) \in [-\ell, \ell]$ , so the oddness of the  $2\ell$ -periodic extension reduces to the oddness of  $\phi_{\text{odd}}$ . This way we obtain an extension to all of  $\mathbb{R}$  which is odd and  $2\ell$ -periodic, and it satisfies

$$\phi_{\text{ext}}(2\ell - x) = \phi_{\text{ext}}(-x) = -\phi_{\text{ext}}(x),$$

where the first equality follows from being  $2\ell$ -periodic, the second from being odd, so  $\phi_{\text{ext}}$  is also odd about  $x = \ell$ .

Now we solve the heat equation on  $\mathbb{R}_x \times (0, \infty)_t$ :

$$\begin{aligned} v_t &= kv_{xx}, \quad x \in \mathbb{R}, \\ v(x, 0) &= \phi_{\text{ext}}(x), \end{aligned}$$

and let  $u$  be the restriction of  $v$  to  $[0, \ell]_x \times [0, \infty)_t$ . Since  $\phi_{\text{ext}}$  is odd both about  $x = 0$  and  $x = \ell$ , we deduce that  $v$  is also such, and thus  $v(0, t) = 0 = v(\ell, t)$ , hence  $u$  satisfies the Dirichlet boundary condition. One can again work out the formula explicitly, but now one will end up with an infinite sum corresponding to the  $2\ell$ -periodic nature of the extension.

Solving the wave equation again works similarly. At the qualitative level, solutions on these intervals, cubes, etc., behave just like the solutions on half spaces, quadrants, etc., so in particular for the heat equation we get smoothness of the solutions, and for the wave equation we get reflected singularities following characteristics.