Then be denoted by $S$ homogeneous PDE and the forcing term are very similar. In fact, let the solution operator of the

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where we let $f_s(x) = f(s, x)$ be the restriction of $f$ to the time $s$ slice. Thus, the solution formula for the inhomogeneous PDE, (1), takes the form

$$u(t, x) = (S(t)\phi)(x) + \int_0^t (S(t-s)f_s)(x) \, ds.$$  

The fact that we can write the solution of the inhomogeneous PDE in terms of the solution of the Cauchy problem for the homogeneous PDE is called Duhamel’s principle.

Physically one may think of (2) as follows. The expression $S(t-s)f_s$ is the solution of the heat equation at time $t$ with initial condition $f(s, x)$ imposed at time $s$. Thus, we think of the forcing as a superposition (namely, integral) of initial conditions given at times $s$ between 0 (when the actual initial condition is imposed) and time $t$ (when the solution is evaluated). Conversely, one could say that the initial condition amounts to a delta-distributational forcing, $\phi(x)\delta_0(t)$, provided that $f$ vanished for time $\leq 0$ and we impose $u(T, x) = 0$ for some $T < 0$ (say, $T = -1$); the delta distribution should be considered as being paired against the characteristic function $[-T, t]$, giving $S(t)\phi$ as a result (at least if $t > 0$).

In fact, this calculation did not depend significantly on particular features of the Laplacian on $\mathbb{R}^n$. That is, suppose that we have another differential operator $L$ on $\mathbb{R}^n$, and consider the PDE

$$\partial_t u - Lu = f, \quad u(0, x) = \phi(x).$$  

Suppose that we can solve the homogeneous problem, i.e. that

$$\partial_t u - Lu = 0, \quad u(0, x) = \phi(x),$$

and let $S(t)$ denote the solution operator, so that $(S(t)\phi)(x) = u(t, x)$, hence

$$(S(0)\phi)(x) = \phi(x),$$

as the initial condition holds, and

$$(\partial_t - L)S(t)\phi = 0,$$

since the PDE holds. We claim that under these assumptions the solution of (3) is given by Duhamel’s formula,

$$u(t, x) = (S(t)\phi)(x) + \int_0^t (S(t-s)f_s)(x) \, ds,$$

just as above. To see this, first note that the initial condition is certainly satisfied, for the integral vanishes if $t = 0$. Next, as $S(t)$ is the solution operator for the homogeneous PDE, $\partial_t - L$ applied to the first term of Duhamel’s formula vanishes. Thus,

$$(\partial_t - L)(S(t)\phi)(x) + \int_0^t (\partial_t - L)(S(t-s)f_s)(x) \, ds \quad = (S(0)f_t)(x) + \int_0^t (\partial_t - L)(S(t-s)f_s)(x) \, ds \quad = (S(0)f_t)(x) + \int_0^t (\partial_t - L)(S(t-s)f_s)(x) \, ds \quad = f(t, x),$$

where the $S(0)f_t$ term arose by differentiating the upper limit of the integral, while the other term by differentiating under the integral sign, and we used that $S(0)f_t = f_t$ and

$$(\partial_t - L)(S(t-s)f_s) = ((\partial_s - L)(S(t-s)f_s))_{s=0} = 0.$$
We can also deal with inhomogeneous boundary conditions. For instance, we can solve

\[ u_t - k\Delta u = f, \quad x_n \geq 0, \]
\[ u(x', 0, t) = h(x', t) \quad \text{(DBC)}, \]
\[ u(x, 0) = \phi(x) \quad \text{(IC)}, \]

(4)

where \( f \) and \( \phi \) are given functions for \( x_n \geq 0 \), and \( h \) is a given function of \((x', t)\). To solve this, we reduce it to a PDE with homogeneous boundary conditions. Thus, let \( F \) be any function on \( \mathbb{R}^n_x \times [0, \infty) \) such that \( F \) satisfies the boundary condition, i.e. \( F(x', 0, t) = h(x', t) \). For instance, we may take \( F(x', 0, t) = h(x', t) \). If \( u \) solves (4) then \( v = u - F \) solves

\[ v_t - k\Delta v = f - (F_t - k\Delta F), \quad x_n \geq 0, \]
\[ v(x', 0, t) = 0 \quad \text{(DBC)}, \]
\[ v(x, 0) = \phi(x) - F(x, 0) \quad \text{(IC)}, \]

(5)

i.e. a PDE with homogeneous boundary condition, but different initial condition and forcing term. Here we used

\[ v_t - k\Delta v = u_t - F_t - k\Delta u + k\Delta F = f - (F_t - k\Delta F). \]

Conversely, if we solve (5), then \( u = v + F \) will solve (4). But we know how to solve (5): either use an appropriate (odd) extension of the data to \( \mathbb{R}^n \) and solve the PDE there, or use Duhamel’s principle and the solution of the homogeneous PDE (with vanishing right hand side) on the half space; see your homework. Thus, (4) can be solved as well.

Similarly, for inhomogeneous Neumann boundary conditions,

\[ u_t - k\Delta u = f, \quad x_n \geq 0, \]
\[ (\partial_{x_n} u)(x', 0, t) = h(x', t) \quad \text{(DBC)}, \]
\[ u(x, 0) = \phi(x) \quad \text{(IC)}, \]

(6)

we take some \( F \) such that \( F \) satisfies the boundary condition, i.e. \((\partial_{x_n} F)(x', 0, t) = h(x', t)\). For instance, we can take

\[ F(x', x_n, t) = x_n h(x', t). \]

Proceeding as above, it suffices to solve

\[ v_t - k\Delta v = f - (F_t - k\Delta F), \quad x_n \geq 0, \]
\[ (\partial_{x_n} v)(x', 0, t) = 0 \quad \text{(NBC)}, \]
\[ v(x, 0) = \phi(x) - F(x, 0) \quad \text{(IC)}, \]

(7)

and let \( u = v + F \). Since (7) has homogeneous boundary conditions, it can be solved as above (either take even extensions, or use Duhamel’s principle on the half-space).

One can also derive a version of Duhamel’s principle for the wave equation, or indeed equations of the form

\[ (\partial_t^2 - L)u = f, \quad u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x). \]

(8)

Let \( S(t) \) be the solution operator corresponding to \( f = 0, \phi = 0 \), i.e. for functions \( \psi, u(t, x) = S(t)\psi(x) \) solves

\[ (\partial_t^2 - L)u = 0, \quad u(x, 0) = 0, \quad u_t(x, 0) = \psi(x), \]
You can solve this by factoring the differential operator on the left hand side as could convert the wave equation into a system; this is done on your problem set.

\[ (\partial_t^2 - L)(S(t)\psi) = 0, \quad S(0)\psi(x) = 0, \quad (\partial_t S(t)\psi)(x)|_{t=0} = \psi(x), \]

Then the solution for (8) is

\[ u(x, t) = \partial_t(S(t)\phi)(x) + (S(t)\psi)(x) + \int_0^t (S(t-s)f_s)(x) \, ds. \]

One checks easily that this indeed solves (8). First, the last term vanishes when \( t = 0 \) (since the integral is from 0 to 0 then) and its \( t \)-derivative at \( t = 0 \) is

\[ S(0)f_0 + \int_0^0 (\partial_t S(t-s)f_s)(x) \, ds = 0, \]

so it does not contribute to the initial conditions. We have \( S(0)\psi = 0 \) and \( \partial_t(S(t)\psi)|_{t=0} = \psi \). Moreover, \( \partial_t(S(t)\phi)|_{t=0} = \phi \) while

\[ \partial_t(\partial_t(S(t)\phi)) = \partial_t^2(S(t)\phi) = L(S(t)\phi), \]

so

\[ \partial_t(\partial_t(S(t)\phi))|_{t=0} = L(S(0)\phi) = 0 \]

since \( S(0)\phi = 0 \). Adding up, we see that all the initial conditions are satisfied. We still need to check that the PDE holds. Certainly \( S(t)\psi \) satisfies the homogeneous PDE, and the same follows for \( \partial_t(S(t)\phi) \) since

\[ (\partial_t^2 - L)\partial_t(S(t)\phi) = \partial_t((\partial_t^2 - L)S(t)\phi) = \partial_t\phi = 0 \]

as \( \partial_t \) commutes with \( \partial_t^2 - L \). It remains to check that the last term of (9) solves the inhomogeneous PDE. First,

\[ \partial_t \left( \int_0^t (S(t-s)f_s)(x) \, ds \right) = S(0)f_t + \int_0^t \partial_t(S(t-s)f_s)(x) \, ds \]

\[ = \int_0^t \partial_t(S(t-s)f_s)(x) \, ds. \]

Thus,

\[ \partial_t^2 \left( \int_0^t (S(t-s)f_s)(x) \, ds \right) = (\partial_t S)(0)f_t(x) + \int_0^t \partial_t^2 S(t-s)f_s(x) \, ds \]

\[ = f(t, x) + \int_0^t L S(t-s)f_s(x) \, ds \]

\[ = f(t, x) + L \left( \int_0^t S(t-s)f_s(x) \, ds \right), \]

so

\[ (\partial_t^2 - L) \left( \int_0^t (S(t-s)f_s)(x) \, ds \right) = f(t, x), \]

as desired. Combining these calculations shows that (9) solves the inhomogeneous PDE.

Of course, (9) here showed up out of the blue. We could have derived this by taking a Fourier transform and solving the inhomogeneous second order ODE

\[ (\partial_t^2 + c^2|\xi|^2)\hat{u}(t, \xi) = \hat{f}(t, \xi), \quad \hat{u}(0, \xi) = \mathcal{F}\phi(\xi), \quad \hat{u}_t(0, \xi) = \mathcal{F}\psi(\xi). \]

You can solve this by factoring the differential operator on the left hand side as \( (\partial_t + ic|\xi|)(\partial_t - ic|\xi|) \), and solving two ODEs. Rather than doing this directly, one could convert the wave equation into a system; this is done on your problem set.
We write out explicitly the solution of the inhomogeneous wave equation on \( \mathbb{R} \):

\[
(\partial^2_t - c^2 \partial^2_x)u = f, \quad u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x).
\]

Then

\[
(S(t)\psi)(x) = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(\sigma) \, d\sigma,
\]

so by (9), the solution of the inhomogeneous PDE is

\[
u(x, t) = \partial_t \left( \frac{1}{2c} \int_{x-ct}^{x+ct} \phi(\sigma) \, d\sigma \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(\sigma) \, d\sigma + \int_0^t \frac{1}{2c} \int_{x-ct(t-s)}^{x+ct(t-s)} f(s, \sigma) \, d\sigma
\]

\[
= \frac{1}{2} (\phi(x + ct) + \phi(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(\sigma) \, d\sigma + \frac{1}{2c} \int_0^t \int_{x-ct(t-s)}^{x+ct(t-s)} f(s, \sigma) \, d\sigma.
\]

The first two terms give d’Alembert’s formula for the homogeneous wave equation. The last term is a constant times the integral of \( f \) over the region

\[
D_{x,t} \cap \{(s, \sigma) : \ s \geq 0 \} = \{(s, \sigma) : \ 0 \leq s \leq t, \ x - c(t-s) \leq \sigma \leq x + c(t-s)\}
\]

\[
= \{(s, \sigma) : \ 0 \leq s \leq t, \ |x - \sigma| \leq c(t-s)\},
\]

i.e. the backward characteristic triangle from \((x,t)\), truncated at the \( x \)-axis (i.e. \( t = 0 \)). Thus, \( u(x, t) \) depends on \( f \) in the part of \( D_{x,t}^- \) after the initial data are imposed, which is another reason why we defined the domain of dependence for the wave equation to be the whole region \( D_{x,t}^- \) (apart from the issue that initial conditions could be imposed at other, non-zero, times \( \leq t \)).

We can again solve the wave equation on half space (and similarly on quadrants, intervals, etc.) with inhomogeneous boundary conditions by the same method as for the heat equation. Thus, consider

\[
(\partial^2_t - c^2 \Delta)u = f, \quad x_n \geq 0,
\]

\[
u(x', 0, t) = h(x', t) \text{ (DBC)},
\]

\[
u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x) \text{ (IC)},
\]

where \( f, \phi, \psi \) are given functions in \( x_n \geq 0 \), \( h \) a given function of \( x' \) and \( t \). Namely, let \( F \) be any function in \( x_n \geq 0 \) such that it satisfies the boundary condition, i.e. \( F(x', 0, t) = h(x', t) \). Then let \( v \) be the solution of the PDE with homogeneous boundary condition (which we can thus solve)

\[
(\partial^2_t - c^2 \Delta)v = f - (\partial^2_t - c^2 \Delta)F, \quad x_n \geq 0,
\]

\[
v(x', 0, t) = 0 \text{ (DBC)},
\]

\[
v(x, 0) = \phi(x) - F(x, 0), \quad u_t(x, 0) = \psi(x) - F_t(x, 0) \text{ (IC)}.
\]

Then \( u = v + F \) solves (11).

One possible choice is of course \( F(x', x_n, t) = h(x', t) \), as above. However, there can be better choices. In particular, suppose that \( n = 1 \). Then any function of the form \( \tilde{g}(x - ct) \), or equivalently of the form \( g(t - \frac{x}{c}) \) solves the wave equation. Now let

\[
g(s) = \begin{cases} 
  h(s), & x > 0, \\
  0, & x < 0,
\end{cases}
\]
Then
\[ F(x, t) = g\left(t - \frac{x}{c}\right) \]
solves the homogeneous wave equation, \( F(0, t) = h(t) \) for \( t \geq 0 \) and \( F(x, 0) = F_t(x, 0) = 0 \) for \( x > 0 \). Thus, the PDE for \( v \) is the ‘same’ as the PDE for \( u \), with the only change that the boundary condition is now homogeneous.

\[
(\partial_t^2 - c^2 \partial_x^2)v = f, \ x \geq 0, \\
v(x', 0, t) = 0 \ (\text{DBC}), \\
v(x, 0) = \phi(x), \ u_t(x, 0) = \psi(x) \ (\text{IC}).
\]

In particular, if \( f = 0, \phi = \psi = 0 \), then the solution of the original PDE, (11), (for \( n = 1 \)) is simply
\[
u(x, t) = g\left(t - \frac{x}{c}\right),
\]
g as in (12). Note that \( u \) is a distributional solution of the PDE even if \( g \) is discontinuous. The latter can happen even if \( h \) is \( C^\infty \), namely if it does not vanish at 0. In general, in view of (12), the smoothness of \( u \) depends not only on the smoothness of \( h \), but on its vanishing at \( t = 0 \), which should be thought of as a matching condition with the initial condition (which vanish!).