Math 205b Homework 5 Solutions

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Problem 1 (R-S IV.40)

Let $B$ denote the unit ball $\{x \in X : ||x|| \leq 1\}$ and $S$ denote the unit sphere. $S \subset B$, so $\overline{S} \subset \overline{B}$ in $X$. Now let $x \in B$. We want to show that any neighborhood of $x$ intersects $S$.

We have a base of the topology around $x$ given by

$$N(\ell_1, \ldots, \ell_n; x; \epsilon) = \{y : |\ell_i(y) - \ell_i(x)| < \epsilon, i = 1, \ldots, n\},$$

where $n \in \mathbb{N}$, $\ell_i \in X^*$, and $\epsilon > 0$. It is thus enough to check that for any $n \in \mathbb{N}$, $\epsilon > 0$, and $\ell_1, \ldots, \ell_n \in X^*$, we have $N(\ell_1, \ldots, \ell_n; x; \epsilon) \cap S \neq \emptyset$. Fix such a neighborhood of $x$ and call it $N$.

$X$ is infinite dimensional, and $\cap_{i=1}^n \ker \ell_i$ has codimension at most $n$, so it contains a nonzero element $y$. For $\alpha \in \mathbb{R}$, consider $x + \alpha y$. $||x|| \leq 1$, so we may choose an $\alpha$ such that $z = x + \alpha y$ has norm 1. Thus $z \in S$, and $\ell_i(z) = \ell_i(x) + \ell_i(y) = \ell_i(x)$, so $z \in N$. Thus $x \in \overline{S}$, proving the claim.

It is not too difficult to see that $B$ is weakly closed. Let $y \notin B$ (so $||y|| > 1$). Hahn-Banach gives us a linear functional $\ell \in X^*$ with $\ell(y) = ||y||$ and $||\ell|| = 1$. Let $\delta = ||y|| / 2$. Then if $x \in B$, we have $|\ell(x)| \leq ||x|| \leq 1$, so $x \notin N(\ell; y; \delta)$ and so $B$ is weakly closed.

Problem 2

$F : X \to Y$ linear, strongly continuous, so if $\ell \in Y^*$, then $\ell \circ F$ is linear and strongly continuous, so $\ell \circ F \in X^*$. Thus $\ell \circ F$ is continuous $(X, \mathcal{T}_w) \to \mathbb{C}$.

We want to show that the inverse image of an open set is open. It is enough to check this on a base for the topology. So, let $y \in Y$, $\ell_1, \ldots, \ell_n \in Y^*$, and $\epsilon > 0$. Consider $N = N(\ell_1, \ldots, \ell_n; y; \epsilon) \subset Y$. Then

$$N = \cap_{i=1}^n \ell_i^{-1}(B(\ell_i(y), \epsilon)) = \cap_{i=1}^n U_i,$$

where $U_i = B(\ell_i(y), \epsilon) \subset \mathbb{C}$ is the open ball of radius $\epsilon$ around $\ell_i(y)$. Then

$$F^{-1}(N) = F^{-1}(\cap_{i=1}^n \ell_i^{-1}(U_i)) = \cap_{i=1}^n F^{-1}(\ell_i^{-1}(U_i)) = \cap_{i=1}^n (\ell_i \circ F)^{-1}(U_i).$$

$\ell_i \circ F$ is continuous, so $(\ell_i \circ F)^{-1}(U_i)$ is open and so $F^{-1}(N)$ is a finite intersection of open sets and thus is open, and so $F : (X, \mathcal{T}_w) \to (Y, \mathcal{U}_w)$ is continuous.
Problem 3

We’ll go ahead and prove the second statement, which then implies the first.

We first need to show that points are closed. Let \( x \in X \). If \( y \in \text{span}\{x\} \), \( y \neq x \), then we may find a linear functional \( \ell \in X^* \) such that \( \ell(x) = 1 \). \( y = \alpha x \) for some \( \alpha \neq 1 \), so we must have \( \ell(y) = \alpha \). If \( \delta = |\alpha - 1|/2 \), then \( x \notin N(\ell; y; \delta) \). If \( y \notin \text{span}\{x\} \), then Hahn-Banach gives us a linear functional \( \ell \in X^* \) such that \( \ell(y) = 0 \) and \( \ell(x) = ||x|| \), so \( x \notin N(\ell; y; ||x||/2) \), so \( X \setminus \{x\} \) is weakly open and thus \( \{x\} \) is weakly closed.

Now let \( x \in X \) and \( C \subset X \) closed with \( x \notin C \). \( C \) closed, so there is a neighborhood of \( x \) disjoint from \( C \). We know a base for \( T \) at \( x \), so we may find \( \ell_1, \ldots, \ell_n \in X^* \) and \( \epsilon > 0 \) such that \( N(\ell_1, \ldots, \ell_n; x; \epsilon) \cap C = \emptyset \). Let \( F_i = C \setminus B(\ell_i(x), \epsilon) \). \( C \) is completely regular (it is a metric space and so is normal), so we may find \( g_i : \mathbb{C} \rightarrow [0, 1] \) such that \( g_i(\ell_i(x)) = 1 \) and \( g_i \equiv 0 \) on \( F_i \). Let \( f_i = g_i \circ \ell_i \), which is continuous \( (X, T) \rightarrow \mathbb{C} \). We then set \( f = \min(f_1, \ldots, f_n) \), which is continuous because it is the minimum of a finite number of continuous functions. Observe that \( f_i(x) = 1 \) for all \( i \), so \( f(x) = 1 \), and if \( y \in C \), then there is some \( i \) such that \( |\ell_i(y) - \ell_i(x)| \geq \epsilon \), so \( \ell_i(y) \in F_i \) and so \( f_i(y) = 0 \). Thus \( f \equiv 0 \) on \( C \), so \( X \) is completely regular (and thus regular).

Problem 4

(1) Suppose that \( x \in \bigcap_{j=1}^{n} \ker f_j \). Then we have, for all \( \alpha \in \mathbb{F} \), \( f_j(\alpha x) = 0 \), so \( |\alpha||f(x)| = |f(\alpha x)| < 1 \). Letting \( |\alpha| \rightarrow \infty \), we must have \( f(x) = 0 \), and so \( x \in \ker f \).

We want to conclude that \( f \) is a linear combination of the \( f_j \). Without loss of generality, we may assume that \( f_1, \ldots, f_n \) are linearly independent. Define \( \Phi : X \rightarrow \mathbb{F}^{n+1} \) by

\[
\Phi(x) = (f_1(x), \ldots, f_n(x), f(x)).
\]

Then \( \Phi \) is linear and \((0, \ldots, 0, 1)\) is not in the image of \( \Phi \). Thus \( \text{Im} \Phi \) is a proper subspace of \( \mathbb{F}^{n+1} \), so there is a nonzero linear functional on \( \mathbb{F}^{n+1} \) that vanishes on \( \text{Im} \Phi \). In other words, there is some \((\alpha_1, \ldots, \alpha_n, \alpha) \in \mathbb{F}^{n+1} \setminus 0 \) such that for all \( x \in X \),

\[
\alpha f(x) + \sum_{j=1}^{n} \alpha_j f_j(x) = 0.
\]

If \( \alpha = 0 \), this gives us a nontrivial linear relation in the \( f_j \), contradicting the assumption that the \( f_j \) were linearly independent, so we have that \( \alpha \neq 0 \). Thus

\[
f(x) = \frac{-1}{\alpha} \sum_{j=1}^{n} \alpha_j f_j(x)
\]

for all \( x \), and so \( f \) is a linear combination of the \( f_j \).
(2) \((X, T)\) is first countable, so there is a countable base \(\{U_k\}\) at 0. We already know one base for \(T\), so for each \(k\), there is some \(n(k) > 0\), \(\epsilon(k) > 0\), and \(f_1^{(k)}, \ldots, f_{n(k)}^{(k)} \in X^*\) such that \(N(f_1^{(k)}, \ldots, f_{n(k)}^{(k)}; 0; \epsilon(k)) \subset U_k\).

We obtain the sequence \(f_j\) by renumbering the countable set

\[
\bigcup_{k=1}^{\infty} \{f_1^{(k)}, \ldots, f_{n(k)}^{(k)}\}.
\]

\(U_k\) is a base, so if \(f \in X^*\), we have some \(k\) such that \(U_k \subset N(f; 0; 1)\). Thus

\[
N(f_1^{(k)}, \ldots, f_{n(k)}^{(k)}; 0; \epsilon(k)) \subset N(f; 0; 1).
\]

In other words, if \(|f_j^{(k)}(x)| < \epsilon(k)\) for \(j = 1, \ldots, n(k)\), then \(|f(x)| < 1\). By part (1), we know that this means \(f\) is a linear combination of the \(f_j^{(k)}\).

(3) We know that \(X^* = X\), so if \((X, T)\) were first countable, we would be able to find (by (2)) a countable set \(\{x_n\}\) in \(X\) such that any element of \(X\) is a finite linear combination of \(\{x_n\}\). We thus want to show that there is no such set.

Though this statement is true in an arbitrary infinite dimensional Banach space (using an argument involving the Baire category theorem), we may see it even more directly in this case. Suppose we have such a set. By applying the Gram-Schmidt process to the \(x_n\), we may assume they are orthonormal. \(X\) is infinite dimensional, so we know that the orthonormal set we obtain is infinite because its span is all of \(X\).

Consider the element \(y = \sum_{n=1}^{\infty} 2^{-n}x_n\). The sum converges because \(X\) is complete and \(\{x_n\}\) is orthonormal.

We then have, by assumption, that \(\sum 2^{-n}x_n = \sum_{n=1}^{N} c_n x_n\) for some \(N\) and \(c_n \in \mathbb{F}\) by our hypothesis on the set \(\{x_n\}\). Then, for any \(k > N\), we have that \(2^{-k} = (y, x_k) = (\sum_{n=1}^{N} c_n x_n, x_k) = 0\), a contradiction.

**Problem 5 (R-S IV.23)**

For any open set \(O \subset \hat{X}\), let \(\hat{O} = O \cap X\). If \(\infty \notin O\), then \(O = \hat{O}\) is open in \(X\). If \(\infty \in O\), then \(\hat{X} \setminus O = X \setminus \hat{O}\) is compact. \(X\) is Hausdorff, so \(X \setminus \hat{O}\) is closed and thus \(\hat{O}\) is open in \(X\).

We now claim that this defines a topology. Clearly \(\emptyset\) and \(\hat{X}\) are open. Now suppose that \(O_i, i \in I\) are open. We divide this up into open sets of the first type \(O_i^1\) that are open sets in \(X\) and those of the second type \(O_i^2\) that contain \(\infty\). If there is no \(O_i^2\) in this collection, then \(\bigcup_{i \in I} O_i = \bigcup_{i \in I} O_i^1\) is open in \(X\) and thus open. If not, then we have \(\hat{X} \setminus \bigcup O_i = (\cap \hat{X} \setminus O_i^1) \cap (\cap \hat{X} \setminus O_i^2)\), which is the intersection of a closed set with a compact one and thus is compact. Finally, we consider \(O_1 \cap O_2\). If both are of the first type, then this is open in \(X\) and thus open in \(\hat{X}\). If not, then \(\hat{X} \setminus (O_1 \cap O_2) = (\hat{X} \setminus O_1) \cup (\hat{X} \setminus O_2)\), which is compact. Thus this defines a topology on \(\hat{X}\).
Problem 6 (Royden, 8.3.24)

(1) $x \mapsto \frac{x}{1+|x|}$ is continuous on $\mathbb{R}$, so $h$ is continuous and $|h| < 1$.

(2) Let $B, C$ be as above, so $B, C \subset A$ are closed, and $A$ is closed, so $B, C$ are closed in $X$. $B \cap C = \emptyset$, so Urysohn’s lemma gives us a continuous function $r$ such that $r = 0$ on $B$ and $r = 1$ on $C$, and $0 \leq r \leq 1$. By letting $h_1 = \frac{2}{3}r - \frac{1}{3}$, we have a continuous function $h_1$ on $X$ such that $h_1 = -\frac{1}{3}$ on $B$, $h_1 = \frac{1}{3}$ on $C$, and $|h(x)| \leq \frac{1}{3}$ for all $x \in X$. Moreover, because $|h(x)| < 1$, we have that if $x \in B$, $|h(x) - h_1(x)| < 2/3$, with a similar statement for $x \in C$. If $x \in A$ but $x \notin B \cup C$, then $|h(x)| < 1/3$, so $|h(x) - h_1(x)| < 2/3$.

(3) By induction (with the inductive step being essentially identical to the argument in (2)), there is a continuous function $h_n$ on $X$ such that $|h_n(x)| \leq 2^{-n-1}/3^n$ for $x \in X$ and $|h(x) - \sum_{i=1}^{n} h_i(x)| < 2^n/3^n$ for all $x \in A$.

(4) Because $|h_n| \leq 2^{-n-1}/3^n$, we have that the series $\sum h_n$ is absolutely summable and thus converges uniformly to a continuous function $k$ on $X$, and $|k| \leq \sum 2^{-n-1}/3^n = 1$. If $x \in A$, then $|h(x) - k(x)| = \lim |h(x) - \sum_{i=1}^{n} h_i(x)| = 0$, so $k = h$ on $A$.

(5) $k$ is continuous, so both $\{x : |k(x)| = 1\}$ and $A$ are closed. $|k| = |h| < 1$ on $A$, so these two sets are disjoint, so the Urysohn lemma gives us a continuous function $\phi$ on $X$ that is 1 on $A$ and 0 on $\{x : |k(x)| = 1\}$.

(6) Let $g = \phi k/(1 - \phi|k|)$. $|k| < 1$ whenever $\phi \neq 0$, so $g$ is the denominator is never 0 and thus $g$ is continuous. Thus $g$ is a continuous function on $X$ such that for $x \in A$,

$$g(x) = \frac{\phi(x)k(x)}{1 - |\phi(x)k(x)|} = \frac{h(x)}{1 - |h(x)|} = \frac{f(x)}{(1 + |f(x)|)(1 - \frac{|f(x)|}{1+|f(x)|})} = f(x).$$