Math 205b Homework 2 Solutions

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Problem 1 (R-S, II.4)

1) For the $\mathbb{R}$ case, we just expand the right hand side and use the symmetry of the inner product:

$$\frac{1}{4} (||x + y||^2 - ||x - y||^2) =$$

$$= \frac{1}{4} ((x, x) + (y, y) + (x, y) + (y, x) - (x, x) - (y, y) + (y, x) + (x, y)) =$$

$$= \frac{1}{2} ((x, y) + (y, x)) = (x, y).$$

For the other case, we again expand the right side, using the relation we just proved:

$$\frac{1}{4} (||x + y||^2 - ||x - y||^2) - \frac{i}{4} (||x + iy||^2 - ||x - iy||^2) =$$

$$= \frac{1}{2} ((x, y) + (y, x)) - \frac{i}{2} ((x, iy) + (iy, x)) =$$

$$= \frac{1}{2} (x, y) + \frac{1}{2} (y, x) - \frac{i^2}{2} (x, y) + \frac{i^2}{2} (y, x) = (x, y).$$

2) If the norm comes from an inner product, then we have

$$||x + y||^2 + ||x - y||^2 =$$

$$= 2(x, x) + 2(y, y) + (x, y) + (y, x) - (x, y) - (y, x) = 2 (||x||^2 + ||y||^2).$$

Now suppose that the norm satisfies the parallelogram law. Assume the field is $\mathbb{C}$ (the argument is similar for $\mathbb{R}$), and define the inner product via the polarization identity from part (1). If $x, y, z \in V$, we have that (writing $x + y = x + \frac{y + z}{2} + \frac{y - z}{2}$ and similar
for \( x + z \):

\[
(x, y) + (x, z) = \frac{1}{4} \left( ||x + y||^2 + ||x + z||^2 - ||x - y||^2 - ||x - z||^2 \right) + \\
- \frac{i}{4} \left( ||x + iy||^2 + ||x + iz||^2 - ||x - iy||^2 - ||x - iz||^2 \right) = \\
\frac{1}{2} \left( ||x + \frac{y + z}{2}||^2 + ||y - z||^2 - ||x - \frac{y + z}{2}||^2 - ||\frac{y - z}{2}||^2 \right) + \\
- \frac{i}{2} \left( ||x + i\frac{y + z}{2}||^2 + ||\frac{y - z}{2}||^2 - ||x - i\frac{y + z}{2}||^2 - ||\frac{y - z}{2}||^2 \right) = \\
\frac{1}{2} \left( ||x + \frac{y + z}{2}||^2 + ||y + z||^2 - ||x - \frac{y + z}{2}||^2 - ||\frac{y - z}{2}||^2 \right) + \\
- \frac{i}{2} \left( ||x + i\frac{y + z}{2}||^2 + ||\frac{y + z}{2}||^2 - ||x - i\frac{y + z}{2}||^2 - ||\frac{y - z}{2}||^2 \right) = \\
= \frac{1}{4} \left( ||x + y + z||^2 + ||x||^2 - ||x - (y + z)||^2 - ||x||^2 \right) + \\
- \frac{i}{4} \left( ||x + i(y + z)||^2 + ||x||^2 - ||x - i(y + z)||^2 - ||x||^2 \right) = (x, y + z).
\]

This holds for all \( x, y, z \), so, in particular, \((x, ny) = n(x, y)\), and so if \( r = p/q \in \mathbb{Q} \), then \((x, ry) = r(x, y)\). Moreover, by the polarization identity,

\[
(x, iy) = \frac{1}{4} \left( ||x + iy||^2 - ||x - iy||^2 \right) + \\
- \frac{i}{4} \left( ||x - y||^2 - ||x + y||^2 \right) = i(x, y).
\]

Combining these two results gives that if \( \alpha \in \mathbb{Q} + i\mathbb{Q} \), then \((x, \alpha y) = \alpha(x, y)\). If \( \alpha \in \mathbb{C} \), then there are \( \alpha_n \in \mathbb{Q} + i\mathbb{Q} \), \( \alpha_n \to \alpha \), so then \( \alpha_n y \to \alpha y \), so \((x, \alpha_n y) \to (x, \alpha y)\) because all the norms above must converge. This gives us that

\[
\alpha(x, y) = \lim \alpha_n(x, y) = \lim (x, \alpha_n y) = (x, \alpha y),
\]

so \((x, y)\) is linear.

Observe that because \( ||i(x - iy)|| = ||x - iy|| \), we have that \((y, x) = (x, y)\), and that

\[
(x, x) = \frac{1}{4} (||2x||^2) - \frac{i}{4} (||1 + i||^2 ||x||^2 - ||1 - i||^2 ||x||^2) = ||x||^2,
\]

so this shows that the norm is induced by \((\cdot, \cdot)\) and that it is also positive definite, so it is an inner product.
(3) Let \( f = \chi_{[0,1/2]}, \ g = \chi_{[1/2,1]} \in L^p(\mathbb{R}) \), so then \( ||f + g||^2 = ||f - g||^2 = 1 \), and \( ||f||^2 = \frac{1}{2} \). Therefore, the parallelogram law does not hold unless \( p = 2 \). (For \( p = \infty \), the same examples work, but the norms are slightly different.)

Let \( a = (1/2,1/2,0,0,\ldots), \ b = (1/2,-1/2,0,0,\ldots) \in \ell_p \), then \( ||a||^2 = \frac{1}{2} + \frac{1}{2} = 1 \), \( ||b||^2 = \frac{1}{2} + \frac{1}{2} = 1 \), and \( ||a + b||^2 = 2 \). The parallelogram law holds.

\[ \text{Problem 2 (R-S, II.8)} \]

Fix \( x \), so (i) and (iii) show \( B(x,\cdot) \) is a continuous linear functional on \( \mathcal{H} \), so there is some \( x' \in \mathcal{H} \) such that \( B(x,y) = (x',y) \) for all \( y \). Let \( Ax = x' \). By construction, for all \( x,y \), we have \( B(x,y) = (Ax,y) \). We want to show that \( A \) is a continuous linear transformation of the right norm and that it is unique. Let \( x,y,z \in \mathcal{H}, \ \alpha \in \mathbb{C} \). Then

\[ (A(x+y),z) = B(x+y,z) = B(x,z) + B(y,z) = (Ax,z) + (Ay,z) = (Ax + Ay,z), \]

holds for all \( z \), so \( A(x+y) = Ax + Ay \). Similarly,

\[ (A(\alpha x),y) = B(\alpha x,y) = \overline{\alpha} B(x,y) = \overline{\alpha}(Ax,y) = (\alpha Ax,y), \]

so \( A(\alpha x) = \alpha Ax \), so \( A \) is linear.

For a fixed \( x \), we choose \( y = Ax \), then \( B(x,Ax) = (Ax,Ax) = ||Ax||^2 \), but \( |B(x,Ax)| \leq C ||x|| ||Ax|| \), so \( ||Ax|| \leq C ||x|| \), so \( A \) is a continuous linear transformation.

We just showed that \( ||A|| \leq C \) for all \( C \) such that \( |B(x,y)| \leq C ||x|| ||y|| \), but we also have that \( |B(x,y)| = |(Ax,y)| \leq ||Ax|| ||y|| \leq ||A|| ||x|| ||y|| \), so \( A \) has the right norm.

Finally, suppose that \( T: \mathcal{H} \to \mathcal{H} \) is another such bounded linear transformation, then

\[ B(x,y) = (Ax,y) = (Tx,y) \]

for all \( x,y \). Then \( A - T \) is a bounded linear transformation, and \( ((A - T)x,y) = 0 \) for all \( x,y \). Choose \( y = (A - T)x \), then \( ||(A - T)x||^2 = 0 \), so \( Ax = Tx \), so \( A \) is unique.

\[ \text{Problem 3} \]

Suppose that \( f \in C^\infty(S^1) \) with \( f(y) = 0 \). We claim that \( (Tf)(y) = 0 \). Indeed, note that

\[ f_1(x) = \frac{f(x)}{y-x} = \int_0^1 f'(ty + (1-t)x)dt \]

is smooth, since we can differentiate with respect to \( x \) under the integral. Then \( f(x) = (e^{iy} - e^{iy})g(x) \) where \( g(x) = \frac{x-y}{e^{ix} - e^{iy}} f_1(x) \) is smooth. Thus \( Tf(y) = (e^{iy} - e^{iy})Tg(y) = 0 \).

We claim now that \( T \) is a multiplication operator. Indeed, consider the constant function \( g(x) = f(y) \), then \( f(x) - g(x) \) vanishes at \( y \), so \( T(f - g)(y) = 0 \), and thus, for each \( y \),

\[ Tf(y) = f(y)T(1)(y) \]
i.e., $Tf(x) = \phi(x)f(x)$ for all $x$, where $\phi(x) = (T1)(x)$. Note that because the constant function $1$ is a smooth function and $T$ takes values in $C^\infty(S^1)$, we must have that $\phi \in C^\infty(S^1)$ as well.

We finally claim that $\phi$ is constant. To do this we use that $T$ commutes with differentiation. Because $\phi(x) = (T1)(x)$, we have

$$\phi'(x) = \frac{d}{dx}(T1)(x) = T\left(\frac{d}{dx}1\right)(x) = 0,$$

so that $\phi$ must be constant. This finishes the proof.

**Problem 4**

1. Integration by parts shows that

$$\langle e_n, f' \rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-inx} f'(x) dx$$

$$= -\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \left(\frac{d}{dx} e^{-inx}\right) f(x) dx + f(x)e^{-inx}\Bigg|_0^{2\pi}$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} i ne^{inx} f(x) dx = in(e_n, f).$$

This holds for all $n$, proving the claim.

2. Let $f \in C^1(S^1)$. Then

$$\sum_{n \in \mathbb{Z}} |(e_n, f)| = |(e_0, f)| + \sum_{n \in \mathbb{Z}, \ n \neq 0} \left| \frac{1}{n} (e_n, f') \right|$$

$$\leq |(e_0, f)| + \left( \sum_{n \neq 0} \frac{1}{n^2} \right)^{\frac{1}{2}} \|f'\|_{L^2}$$

$$\leq C \|f\|_{C^0} + C \|f'\|_{L^2} \leq C \|f\|_{C^1(S^1)},$$

which proves $\mathcal{F}f \in \ell_1(\mathbb{Z})$ and $\|\mathcal{F}f\|_{\ell_1(\mathbb{Z})} \leq C \|f\|_{C^1(S^1)}$ so that $\mathcal{F}$ is continuous.

3. We start by noting that $\|e_n\|_{C^0(S^1)} = \frac{1}{\sqrt{2\pi}}$ for all $n$. This implies that for $\{a_n\} \in \ell_1(\mathbb{Z})$, the series $\sum a_n e_n$ converges absolutely (and so uniformly), as

$$\left| \sum a_n e_n(x) \right| \leq \frac{1}{\sqrt{2\pi}} \|\{a_n\}\|_{\ell_1(\mathbb{Z})}.$$

Note that this inequality also shows that the map is continuous. Together with the previous part, we may conclude that the Fourier series of $f \in C^1(S^1)$ converges uniformly.
4. Note that the previous part shows that both \( f = \mathcal{F}^{-1}\{c_n\} \) and \( g = \mathcal{F}^{-1}\{inc_n\} \) are continuous functions, defined by uniformly convergent series. Note further that for each partial sum \( f_N = \sum_{n=-N}^{N} c_n e_n \), we have that \( f'_N = \sum_{n=-N}^{N} inc_n e_2 \). We know already that the sequence \( f'_N \) converges uniformly to \( \mathcal{F}^{-1}\{inc_n\} \). We claim now that this is enough to show that \( f' = g \) and so that \( f \in C^1(\mathbb{S}^1) \). Indeed, we observe that for \( x \in [0, 2\pi] \),

\[
 f(x) = \lim_{N \to \infty} f_N(x) = \lim_{N \to \infty} \int_0^x f'_N(y) dy.
\]

\( f'_N \) converges uniformly to \( g \) and \( \mathbb{S}^1 \) is compact, so we may conclude by the dominated convergence theorem that \( f(x) = \int_0^x g(y) dy \). In particular, the fundamental theorem of calculus then tells us that \( f'(x) = g(x) \) and so \( f \in C^1(\mathbb{S}^1) \). This also shows that \( \frac{d}{dx}\mathcal{F}^{-1}\{a_n\} = \mathcal{F}^{-1}\{ina_n\} \).

5. We start by calculating, for \( f \in L^2(\mathbb{S}^1) \),

\[
 \mathcal{F}\left(e^{ix} f\right)_n = (e_n, e^{ix} f) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-inx} e^{ix} f(x) dx
 = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-i(n-1)x} f(x) dx = (e_{n-1}, f) = (\mathcal{F} f)_{n-1}.
\]

If \( \{c_n\} \in \ell_2(\mathbb{Z}) \), then

\[
 \mathcal{F}^{-1}\{c_n\} = \frac{1}{\sqrt{2\pi}} \sum c_{n-1} e^{inx} = \frac{e^{ix}}{\sqrt{2\pi}} \sum c_{n-1} e^{i(n-1)x}
 = e^{i\pi} \frac{1}{\sqrt{2\pi}} \sum c_n e^{inx} = e^{ix} \mathcal{F}^{-1}\{c_n\}.
\]

6. Part 2. of this problem shows that if \( f \) is in \( C^1 \), then \( \mathcal{F}f \) lies in \( \ell_1 \). In particular if \( f \in C^\infty(\mathbb{S}^1) \), then \( f^{(k)} \in C^1(\mathbb{S}^1) \) for any \( k \) and so \( \{(in)^k(\mathcal{F} f)_n\} \in \ell_1(\mathbb{Z}) \) for any \( k \geq 0 \). In particular, this is bounded and so \( \mathcal{F}f \in s(\mathbb{Z}) \).

Similarly, part 4. of this problem shows that if \( \{nc_n\} \in \ell_1(\mathbb{Z}) \) then \( \mathcal{F}^{-1}\{c_n\} \in C^1(\mathbb{S}^1) \). If \( \{c_n\} \in s(\mathbb{Z}) \), then \( \{n(in)^k c_n\} \in \ell_1(\mathbb{Z}) \) for any \( k \). Thus \( \mathcal{F}^{-1}\{(in)^k\} = \frac{d^k}{dx^k} \mathcal{F}^{-1}\{c_n\} \), and so \( \mathcal{F}^{-1}\{c_n\} \in C^{k+1}(\mathbb{S}^1) \) for any \( k \), i.e., in \( C^\infty(\mathbb{S}^1) \).

7. By parts 1., 4., 5., and 6. of this problem, we know that \( \mathcal{F}^{-1} \mathcal{F} : C^\infty(\mathbb{S}^1) \to C^\infty(\mathbb{S}^1) \) commutes with multiplication by \( e^{ix} \) and with differentiation. By problem 4., we conclude that \( \mathcal{F}^{-1} \mathcal{F} = c \text{Id} \), where \( c \) is a constant. To determine the value of \( c \), we consider the function 1.

\[
 c_n = (\mathcal{F} f)_n = (e_n, 1) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-inx} = \begin{cases} \sqrt{2\pi} & n = 0 \\ 0 & n \neq 0 \end{cases}.
\]
We then observe that $\mathcal{F}^{-1} \mathcal{F}1 = \sqrt{2\pi} \, e_0 = 1$, so that $c = 1$.

8. $\mathcal{F}^{-1} \mathcal{F} = \text{Id}$ on $C^\infty(\mathbb{S}^1)$ and is continuous on $L^2$. We know that $C^\infty(\mathbb{S}^1)$ is dense in $L^2(\mathbb{S}^1)$, so it must equal the identifc on $L^2$. 