Problem 1 (R-S, II.4)

(1) For the \( \mathbb{R} \) case, we just expand the right hand side and use the symmetry of the inner product:

\[
\frac{1}{4} (||x + y||^2 - ||x - y||^2) =
\]

\[
= \frac{1}{4} ((x, x) + (y, y) + (x, y) + (y, x) - (x, x) - (y, y) + (x, y) + (y, x)) =
\]

\[
= \frac{1}{2} ((x, y) + (y, x)) = (x, y).
\]

For the other case, we again expand the right side, using the relation we just proved:

\[
\frac{1}{4} (||x + y||^2 - ||x - y||^2) - \frac{i}{4} (||x + iy||^2 - ||x - iy||^2) =
\]

\[
= \frac{1}{2} ((x, y) + (y, x)) - \frac{i}{2} ((x, iy) + (iy, x)) =
\]

\[
= \frac{1}{2} (x, y) + \frac{1}{2} (y, x) - \frac{i^2}{2} (x, y) + \frac{i^2}{2} (y, x) = (x, y).
\]

(2) If the norm comes from an inner product, then we have

\[
||x + y||^2 + ||x - y||^2 =
\]

\[
= 2(x, x) + 2(y, y) + (x, y) + (y, x) - (x, y) - (y, x) = 2 \left(||x||^2 + ||y||^2\right).
\]

Now suppose that the norm satisfies the parallelogram law. Assume the field is \( \mathbb{C} \) (the argument is similar for \( \mathbb{R} \)), and define the inner product via the polarization identity from part (1). If \( x, y, z \in V \), we have that (writing \( x + y = x + \frac{y+z}{2} + \frac{y-z}{2} \) and similar
for \( x + z \):

\[
(x, y) + (x, z) = \frac{1}{4} \left( ||x + y||^2 + ||x + z||^2 - ||x - y||^2 - ||x - z||^2 \right) + \\
-\frac{i}{4} \left( ||x + iy||^2 + ||x + iz||^2 - ||x - iy||^2 - ||x - iz||^2 \right) = \\
\frac{1}{2} \left( ||x + \frac{y + z}{2}||^2 - \frac{1}{2} \left( ||x - \frac{y + z}{2}||^2 - ||y - z||^2 \right) \right) + \\
-\frac{i}{2} \left( ||x + \frac{i(y + z)}{2}||^2 - \frac{1}{2} \left( ||x - \frac{i(y + z)}{2}||^2 - ||y - z||^2 \right) \right) = \\
\frac{1}{2} \left( ||x + \frac{y + z}{2}||^2 - \frac{1}{2} \left( ||x - \frac{y + z}{2}||^2 - ||y - z||^2 \right) \right) + \\
-\frac{i}{2} \left( ||x + \frac{i(y + z)}{2}||^2 - \frac{1}{2} \left( ||x - \frac{i(y + z)}{2}||^2 - ||y - z||^2 \right) \right) = \\
\frac{1}{4} \left( ||x + y + z||^2 + ||x||^2 - ||x - (y + z)||^2 - ||x||^2 \right) + \\
-\frac{i}{4} \left( ||x + i(y + z)||^2 + ||x||^2 - ||x - i(y + z)||^2 - ||x||^2 \right) = (x, y + z).
\]

This holds for all \( x, y, z \), so, in particular, \((x, ny) = n(x, y)\), and so if \( r = p/q \in \mathbb{Q}\), then \((x, ry) = r(x, y)\). Moreover, by the polarization identity,

\[
(x, iy) = \frac{1}{4} \left( ||x + iy||^2 - ||x - iy||^2 \right) + \\
-\frac{i}{4} \left( ||x - y||^2 - ||x + y||^2 \right) = i(x, y).
\]

Combining these two results gives that if \( \alpha \in \mathbb{Q} + i\mathbb{Q} \), then \((x, \alpha y) = \alpha(x, y)\). If \( \alpha \in \mathbb{C} \), then there are \( \alpha_n \in \mathbb{Q} + i\mathbb{Q}, \alpha_n \to \alpha \), so then \( \alpha_n y \to \alpha y \), so \((x, \alpha_n y) \to (x, \alpha y)\) because all the norms above must converge. This gives us that

\[
\alpha(x, y) = \lim \alpha_n(x, y) = \lim (x, \alpha_n y) = (x, \alpha y),
\]

so \((x, y)\) is linear.

Observe that because \( ||i(x - iy)|| = ||x - iy|| \), we have that \((y, x) = (x, y)\), and that

\[
(x, x) = \frac{1}{4}(||2x||^2) - \frac{i}{4}(||1 + i||^2 ||x||^2 - ||1 - i||^2 ||x||^2) = ||x||^2,
\]

so this shows that the norm is induced by \((\cdot, \cdot)\) and that it is also positive definite, so it is an inner product.
Problem 3

(3) Let \( f = \chi_{[0,1/2]} \), \( g = \chi_{[1/2,1]} \) \( L^p(\mathbb{R}) \), so then \( \|f + g\|^2 = \|f - g\|^2 = 1 \), and \( \|f\|^2 = \|g\|^2 = \left(\frac{1}{2}\right)^{2/p} \), so the parallelogram law does not hold unless \( p = 2 \). (For \( p = \infty \), the same examples work, but the norms are slightly different.)

Let \( a = (1/2, 1/2, 0, 0, \ldots) \), \( b = (1/2, -1/2, 0, 0, \ldots) \) \( \ell_p \), then \( \|a\|^2 = \|b\|^2 = \left(\frac{1}{2}\right)^{2/p} + \left(\frac{1}{2}\right)^{2/p} \), while \( \|a + b\|^2 = \|a - b\|^2 = 1 \), so the parallelogram law does not hold unless \( p = 2 \).

Problem 2 (R-S, II.8)

Fix \( x \), so (i) and (iii) show \( B(x, \cdot) \) is a continuous linear functional on \( \mathcal{H} \), so there is some \( x' \in \mathcal{H} \) such that \( B(x, y) = (x', y) \) for all \( y \). Let \( Ax = x' \). By construction, for all \( x, y \), we have \( B(x, y) = (Ax, y) \). We want to show that \( A \) is a continuous linear transformation with the right norm and that it is unique. Let \( x, y, z \in \mathcal{H} \), \( \alpha \in \mathbb{C} \). Then

\[
(A(x + y), z) = B(x + y, z) = B(x, z) + B(y, z) = (Ax, z) + (Ay, z) = (Ax + Ay, z),
\]

holds for all \( z \), so \( A(x + y) = Ax + Ay \). Similarly,

\[
(A(\alpha x), y) = B(\alpha x, y) = \bar{\alpha} B(x, y) = \bar{\alpha} (Ax, y) = (\alpha Ax, y),
\]

so \( A(\alpha x) = \alpha Ax \), so \( A \) is linear.

For a fixed \( x \), we choose \( y = Ax \), then \( B(x, Ax) = (Ax, Ax) = \|Ax\|^2 \), but \( |B(x, Ax)| \leq C \|x\| \|Ax\| \), so \( \|Ax\| \leq C \|x\| \), so \( A \) is a continuous linear transformation.

We just showed that \( \|A\| \leq C \) for all \( C \) such that \( |B(x, y)| \leq C \|x\| \|y\| \), but we also have that \( |B(x, y)| = |(Ax, y)| \leq \|Ax\| \|y\| \leq |A| \|x\| \|y\| \), so \( A \) has the right norm.

Finally, suppose that \( T : \mathcal{H} \to \mathcal{H} \) is another such bounded linear transformation, then

\[
B(x, y) = (Ax, y) = (Tx, y)
\]

for all \( x, y \). Then \( A - T \) is a bounded linear transformation, and \( ((A - T)x, y) = 0 \) for all \( x, y \). Choose \( y = (A - T)x \), then \( \|(A - T)x\|^2 = 0 \), so \( Ax = Tx \), so \( A \) is unique.

Problem 3

Suppose that \( f \in C^\infty(\mathbb{S}^1) \) with \( f(y) = 0 \). We claim that \( (Tf)(y) = 0 \). Indeed, note that

\[
f_1(x) = \frac{f(x)}{y - x} = \int_0^1 f'(ty + (1 - t)x) dt
\]

is smooth, since we can differentiate with respect to \( x \) under the integral. Then \( f(x) = (e^{ix} - e^{iy}) \), where \( g(x) = \frac{x-y}{e^{ix} - e^{iy}} \) \( f_1(x) \) is smooth. Thus \( Tf(y) = (e^{iy} - e^{iy})Tg(y) = 0 \).

We claim now that \( T \) is a multiplication operator. Indeed, consider the constant function \( g(x) = f(y) \), then \( f(x) - g(x) \) vanishes at \( y \), so \( T(f - g)(y) = 0 \), and thus, for each \( y \),

\[
Tf(y) = f(y)T(1)(y),
\]

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i.e., $Tf(x) = \phi(x)f(x)$ for all $x$, where $\phi(x) = (T1)(x)$. Note that because the constant function 1 is a smooth function and $T$ takes values in $C^\infty(S^1)$, we must have that $\phi \in C^\infty(S^1)$ as well.

We finally claim that $\phi$ is constant. To do this we use that $T$ commutes with differentiation. Because $\phi(x) = (T1)(x)$, we have

$$\phi'(x) = \frac{d}{dx}(T1)(x) = T\left(\frac{d}{dx}1\right)(x) = 0,$$

so that $\phi$ must be constant. This finishes the proof.

**Problem 4**

1. Integration by parts shows that

$$\langle e_n, f' \rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-inx} f'(x) dx$$

$$= -\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \left(\frac{d}{dx}e^{-inx}\right) f(x) dx + f(x)e^{-inx}|_0^{2\pi}$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} ine^{-inx} f(x) dx = in(e_n, f).$$

This holds for all $n$, proving the claim.

2. Let $f \in C^1(S^1)$. Then

$$\sum_{n \in \mathbb{Z}} |(e_n, f)| = |(e_0, f)| + \sum_{n \in \mathbb{Z}, n \neq 0} \left|\frac{1}{n}(e_n, f')\right|$$

$$\leq |(e_0, f)| + \left(\sum_{n \neq 0} \frac{1}{n^2}\right)^{\frac{1}{2}} \|f'\|_{L^2}$$

$$\leq C \|f\|_{C^0} + C \|f'\|_{L^2} \leq C \|f\|_{C^1(S^1)} \cdot$$

which proves $\mathcal{F}f \in \ell_1(\mathbb{Z})$ and $\|\mathcal{F}f\|_{\ell_1(\mathbb{Z})} \leq C \|f\|_{C^1(S^1)}$ so that $\mathcal{F}$ is continuous.

3. We start by noting that $\|e_n\|_{C^0(S^1)} = \frac{1}{\sqrt{2\pi}}$ for all $n$. This implies that for $\{a_n\} \in \ell_1(\mathbb{Z})$, the series $\sum a_n e_n$ converges absolutely (and so uniformly), as

$$\left|\sum a_n e_n(x)\right| \leq \frac{1}{\sqrt{2\pi}} \|\{a_n\}\|_{\ell_1(\mathbb{Z})}.$$

Note that this inequality also shows that the map is continuous. Together with the previous part, we may conclude that the Fourier series of $f \in C^1(S^1)$ converges uniformly.
4. Note that the previous part shows that both $f = F^{-1}\{c_n\}$ and $g = F^{-1}\{inc_n\}$ are continuous functions, defined by uniformly convergent series. Note further that for each partial sum $f_N = \sum_{n=-N}^{N} c_n e_n$, we have that $f'_N = \sum_{n=-N}^{N} inc_n e_{2n}$. We know already that the sequence $f'_N$ converges uniformly to $F^{-1}\{inc_n\}$. We claim now that this is enough to show that $f' = g$ and so that $f \in C^1(S^1)$. Indeed, we observe that for $x \in [0, 2\pi]$,

$$f(x) = \lim_{N \to \infty} f(x) = \lim_{N \to \infty} \int_0^x f'_N(y) dy.$$ 

$f'_N$ converges uniformly to $g$ and $S^1$ is compact, so we may conclude by the dominated convergence theorem that $f(x) = \int_0^x g(y) dy$. In particular, the fundamental theorem of calculus then tells us that $f'(x) = g(x)$ and so $f \in C^1(S^1)$. This also shows that $\frac{d}{dx} F^{-1}\{a_n\} = F^{-1}\{ina_n\}$.

5. We start by calculating, for $f \in L^2(S^1)$,

$$F(e^{ix}f) = (e_n, e^{ix}f) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-inx} e^{ix} f(x) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-i(n-1)x} f(x) dx = (e_{n-1}, f) = (Ff)_{n-1}.$$

If $\{c_n\} \in \ell_2(\mathbb{Z})$, then

$$F^{-1}\{c_{n-1}\} = \frac{1}{\sqrt{2\pi}} \sum c_{n-1} e^{inx} = \frac{e^{ix}}{\sqrt{2\pi}} \sum c_{n-1} e^{i(n-1)x}$$

$$= e^{ix} \frac{1}{\sqrt{2\pi}} \sum c_{n} e^{inx} = e^{ix} \sum c_{n} e^{inx} = e^{ix} F^{-1}\{c_n\}.$$

6. Part 2. of this problem shows that if $f$ is in $C^1$, then $Ff$ lies in $\ell_1$. In particular if $f \in C^\infty(S^1)$, then $f^{(k)} \in C^1(S^1)$ for any $k$ and so $\{(in)^k(Ff)_n\} \in \ell_1(\mathbb{Z})$ for any $k \geq 0$. In particular, this is bounded and so $Ff \in s(\mathbb{Z})$.

Similarly, part 4. of this problem shows that if $\{nc_n\} \in \ell_1(\mathbb{Z})$ then $F^{-1}\{c_n\} \in C^1(S^1)$. If $\{c_n\} \in s(\mathbb{Z})$, then $\{n(in)^k c_n\} \in \ell_1(\mathbb{Z})$ for any $k$. Thus $F^{-1}\{(in)^k c_n\} \in C^1(S^1)$ for any $k$. By iterating problem part 4., we see that $F^{-1}\{(in)^k\} = \frac{d^k}{dx^k} F^{-1}\{c_n\}$, and so $F^{-1}\{c_n\} \in C^{k+1}(S^1)$ for any $k$, i.e., in $C^\infty(S^1)$.

7. By parts 1., 4., 5., and 6. of this problem, we know that $F^{-1}F : C^\infty(S^1) \to C^\infty(S^1)$ commutes with multiplication by $e^{ix}$ and with differentiation. By problem 4., we conclude that $F^{-1}F = cI$, where $c$ is a constant. To determine the value of $c$, we consider the function $1$.

$$c_n = (F1)_n = (e_n, 1) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-inx} = \begin{cases} \sqrt{2\pi} & n = 0 \\ 0 & n \neq 0. \end{cases}$$
We then observe that $F^{-1}F1 = \sqrt{2\pi} e_0 = 1$, so that $c = 1$.

8. $F^{-1}F = \text{Id}$ on $C^\infty(S^1)$ and is continuous on $L^2$. We know that $C^\infty(S^1)$ is dense in $L^2(S^1)$, so it must equal the identify on $L^2$. 
