Problem 1 (R-S, I.3) Let $\epsilon > 0$. $x_n$ is a Cauchy sequence, so there is some $k_1$ such that for all $m, n > k_1$, $\rho(x_m, x_n) < \frac{\epsilon}{2}$. $x_n(i) \to x_\infty$, so there is some $k_2$ such that for all $i \geq k_2$, $\rho(x_{n(i)}, x_\infty) < \frac{\epsilon}{2}$.

Let $k = \max(k_1, k_2)$. Then, because $n(i) \geq i$ for all $i$, we have that $\rho(x_{n(i)}, x_\infty) < \frac{\epsilon}{2}$.

Now, if we have any $n > k$, then

$$\rho(x_n, x_\infty) \leq \rho(x_n, x_{n(k)}) + \rho(x_{n(k)}, x_\infty) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$ 

Problem 2 (R-S, I.4) Suppose not, so then there is some $\epsilon > 0$ such that for all $k$, there is an $n > k$ with $\rho(x_n, x_\infty) \geq \epsilon$.

We may thus inductively choose a subsequence $x_{n(i)}$ such that $\rho(x_{n(i)}, x_\infty) \geq \epsilon$ for all $i$. This sequence is bounded away from $x_\infty$, so it cannot have a subsequence converging to $x_\infty$, a contradiction. Thus $x_n \to x_\infty$.

Problem 3 (R-S, I.5) We consider Cauchy sequences in $M$. If $x_n$ and $y_n$ are two Cauchy sequences we say that $x_n$ is equivalent to $y_n$ ($x_n \sim y_n$) if $\lim_{n \to \infty} d(x_n, y_n) = 0$. We claim this is an equivalence relation. $d(x, x) = 0$ for all $x$, and $d$ is symmetric, so this relation is reflexive and symmetric. Moreover, if $(x_n) \sim (y_n)$ and $(y_n) \sim (z_n)$, then the triangle inequality gives

$$\lim_{n \to \infty} d(x_n, z_n) \leq \lim_{n \to \infty} (d(y_n, z_n) + d(x_n, y_n)) = 0,$$

so the relation is transitive and thus an equivalence relation.

We now define $\tilde{M}$ to be the set of equivalence classes of Cauchy sequences in $M$. If $x_n$ and $y_n$ are two such sequences, then

$$|d(x_n, y_n) - d(x_m, y_m)| \leq |d(x_n, y_n) - d(x_m, y_n)| + |d(x_m, y_n) - d(x_m, y_m)| \leq |d(x_n, x_m)| + |d(y_n, y_m)|,$$

so $d(x_n, y_n)$ is a Cauchy sequence, so it converges to a limit in $\mathbb{R}$. We define the distance between $[x_n]$ and $[y_n]$ (the equivalence classes represented by $x_n$ and $y_n$, respectively) in $\tilde{M}$
to be this limit. We still have to show that $\tilde{d}([x_n], [y_n])$ is well-defined. If $x'_n$ and $y'_n$ are are two other representatives for $[x_n]$ and $[y_n]$, then

$$\left| \lim_{n \to \infty} (d(x'_n, y'_n) - d(x_n, y_n)) \right| \leq \lim_{n \to \infty} (|d(x'_n, y'_n) - d(x'_n, y_n)| + |d(x'_n, y_n) - d(x_n, y_n)|) \leq \lim_{n \to \infty} (d(y'_n, y_n) + d(x'_n, x_n)) = 0,$$

so the limit is well-defined.

Now we claim that $\tilde{d}$ is a metric: $\tilde{d}$ is clearly positive, and if $\tilde{d}([x_n], [y_n]) = 0$ then $[x_n] = [y_n]$ by the definition of our equivalence relation, so $\tilde{d}$ is positive definite. $\tilde{d}$ is symmetric because $d$ is symmetric. $\tilde{d}$ obeys the triangle inequality because $d$ does:

$$\tilde{d}([x_n], [y_n]) = \lim_{n \to \infty} d(x_n, y_n) \leq \lim_{n \to \infty} d(x_n, z_n) + d(z_n, y_n) = \tilde{d}([x_n], [z_n]) + \tilde{d}([z_n], [y_n]),$$

and so $\tilde{d}$ is a metric.

Now we let $\iota : M \to \tilde{M}$ be given by $x \mapsto [x]$, where $[x]$ is represented by the constant sequence $x_n = x$. $\tilde{d}(\iota(x), \iota(y)) = \lim d(x, y) = d(x, y)$, so $\iota$ is an isometry.

We now claim that $\iota(M)$ is dense in $\tilde{M}$. Let $[x_n] \in \tilde{M}$ and let $\epsilon > 0$. $x_n$ is Cauchy, so there is some $N$ such that if $m, n \geq N$ then $d(x_n, x_m) < \epsilon$, so then if we let $x = x_N$, we have that $\tilde{d}(\iota(x), [x_n]) = \lim d(x_N, x_n) < \epsilon$, so $\iota(M)$ is dense in $\tilde{M}$.

If $x_n$ is a Cauchy sequence in $M$, then $\iota(x_n) \to [x_n]$ in $M$:

$$\tilde{d}(\iota(x_N), [x_n]) = \lim_{n \to \infty} d(x_N, x_n),$$

which tends to 0 as $N \to \infty$ because $x_n$ is Cauchy. $\iota(M)$ is dense in $\tilde{M}$, so if $[x_n]^k$ is Cauchy in $\tilde{M}$, then we claim there is a Cauchy sequence $\iota(y^k)$ in $\iota(M)$ such that $\tilde{d}([x_n]^k, \iota(y^k)) \to 0$. Indeed, by density, for each $k$, there is an element $y^k \in M$ such that $\tilde{d}([x_n]^k, \iota(y^k)) < \frac{1}{k}$. $\iota(y^k)$ is Cauchy because

$$\tilde{d}(\iota(y^k), \iota(y^j)) \leq \tilde{d}(\iota(y^k), [x_n]^k) + \tilde{d}([x_n]^k, [x_n]^j) + \tilde{d}([x_n]^j, \iota(y^j)).$$

We thus know that $\iota(y^k)$ converges in $\tilde{M}$ and so $[x_n]^k$ also converges in $\tilde{M}$ by our condition on $\iota(y^k)$.

We must finally show that $\tilde{M}$ is essentially unique. For $x \in M$, we let $j(\iota(x)) = \iota'(x)$ in $\tilde{M}'$. $\tilde{d}'(j(\iota(x)), j(\iota(y))) = d(x, y) = \tilde{d}(\iota(x), \iota(y))$, so $j$ is an isometry on $\iota(M)$. If $x \in \tilde{M}$, there is a sequence $x_n$ in $\iota(M)$ such that $x_n \to x$ by the density of $\iota(M)$. We then have that

$$\tilde{d}'(j(x_n), j(x_m)) = \tilde{d}(x_n, x_m) \to 0,$$

so $j(x_n)$ is a Cauchy sequence in $\tilde{M}'$, and so has a limit which we’ll call $j(x)$. $j(x)$ is well-defined because if $x_n, y_n \in \iota(M)$ both converge to $x$, then $\tilde{d}'(j(x_n), j(y_n)) = d(x_n, y_n) \to 0$. $j$ is an isometry because we have that

$$\tilde{d}(j(x), j(y)) = \lim \tilde{d}'(j(x_n), j(y_n)) = \lim \tilde{d}(x_n, y_n) = \tilde{d}(x, y).$$
We now must only show that \( j \) is invertible. \( j \) is injective (and continuous) because it is an isometry. If \( j \) is also surjective then its inverse will be continuous because its inverse will also be an isometry. Let \( z \in M' \). \( \iota'(M) \) is dense in \( M' \), so we can take \( z_n \in \iota'(M) \) with \( z_n \to z \). \( z_n \in \iota'(M) \), so \( z_n = \iota'(x_n) = j(\iota(x_n)) = j(y_n) \) for \( y_n \in \iota(M) \). \( z_n \to z \), so \( z_n \) is Cauchy, so \( y_n \) is also Cauchy because \( j \) is an isometry, and so \( y_n \to y \in M \), and

\[
\bar{d}(j(y), z) = \lim_{n} \bar{d}(j(y_n), z) = \lim_{n,m} \bar{d}(j(y_n), j(y_m)) = 0,
\]

so \( j \) is surjective and hence invertible.

**Problem 4 (R-S, I.7)**

(4) implies (3): If \( T \) is bounded, then \( ||Tx|| \leq C ||x|| \) for all \( x \), so \( ||Tx - Ty|| \leq C ||x - y|| \), so \( T \) is uniformly continuous.

(3) implies (2): This is true by definition.

(2) implies (1): Also true by definition.

(1) implies (4): Now suppose that \( T \) is continuous at \( x_0 \), so for all \( \epsilon > 0 \), there is some \( \delta > 0 \) such that \( ||x - x_0|| < \delta \) implies \( ||Tx - Tx_0|| < \epsilon \). Then if \( ||y|| < \delta \) we have \( ||Ty|| = ||T(y + x_0) - Tx_0|| < \epsilon \), so we may assume \( x_0 = 0 \).

If we let \( \epsilon = 1 \), we have that \( ||y|| < \delta \) implies \( ||Ty|| < 1 \). So, if we take an arbitrary \( x \neq 0 \) (\( T \) is linear so \( T(0) = 0 \)) and we consider \( y = \frac{\delta}{2||x||}x \), so \( ||y|| = \delta/2 < \delta \), so \( ||Ty|| < 1 \). So, \( ||Tx|| = \frac{2||x||}{\delta}||Ty|| \leq \frac{2}{\delta} ||x|| \), so \( T \) is bounded.

**Problem 5 (R-S, I.32)** \([0,1] \times [0,1] \) is compact, so \( F \) is uniformly continuous, so for all \( \epsilon > 0 \) there is some \( \delta > 0 \) such that if \( |x_1 - x_2| + |y_1 - y_2| < \delta \) then \( |F(x_1, y_1) - F(x_2, y_2)| < \epsilon \).

Then if \( \epsilon > 0 \), take \( \delta > 0 \) as above, and if \( |x_1 - x_2| < \delta \),

\[
|F(x_1) - F(x_2)| = \left| \int_0^1 (F(x_1, y) - F(x_2, y)) f(y) \, dy \right| \leq \int_0^1 |F(x_1, y) - F(x_2, y)| f(y) \, dy \leq \int_0^1 \epsilon \, dy = \epsilon,
\]

so the family is equicontinuous.

Arzela-Ascoli (since \( ||f_n|| \leq 1 \) gives the uniform bound) then gives the convergent subsequence.

**Problem 6 (R-S, II.1)**

(1) We have that

\[
|(x_n, y_n) - (x_m, y_m)| \leq |(x_n - x_m, y_n)| + |(x_m, y_n - y_m)| \leq ||x_n - x_m|| ||y_n|| + ||x_m|| ||y_n - y_m|| \leq \epsilon (||y|| + 1) + \epsilon (||x|| + 1)
\]

3
for large enough $n, m$, so $(x_n, y_n)$ is Cauchy and must converge. This is well defined: Suppose $x'_n \to x$ and $y'_n \to y$, then

$$|(x_n, y_n) - (x'_n, y'_n)| \leq |(x_n - x'_n, y_n)| + |(x'_n, y_n - y'_n)| \to 0.$$  

This is the right norm, too:

$$||x||^2 = \lim ||x_n||^2 = \lim (x_n, x_n) = (x, x).$$  

We finally check that this satisfies the correct properties. $(\cdot, \cdot)$ is bilinear (or sesqui-linear if we’re working over $\mathbb{C}$) because limits are linear and the inner product is. It is symmetric (or conjugate-symmetric) and positive for the same reason. $(x, x) = 0$ if and only if $(x_n, x_n) \to 0$, so $x_n \to 0$, so $x = 0$, so it is positive-definite. Thus $(\cdot, \cdot)$ extends to $\bar{V}$.

(2) For each $x \in V$, $(\cdot, \cdot)$ is a bounded linear functional. $\mathbb{C}$ is complete, so we may extend it to an element $f_x$ of $\bar{V}^*$ by the BLT theorem, and $f_x$ is bounded by $||x||$. We now consider the map $V \to \bar{V}^*$ given by $x \mapsto \bar{f}_x$ (here the bar denotes complex conjugate). In order to use the BLT theorem again, we need to show that this map is linear. Suppose $\alpha \in \mathbb{C}, x, y, z \in V$. $\bar{f}_{\alpha x}(y) = (\alpha x, y) = \alpha (x, y) = \alpha \bar{f}_x(y)$, so $\bar{f}_{\alpha x}$ and $\bar{f}_x$ agree on $V$. Similarly, $\bar{f}_{x+z}$ and $\bar{f}_x + \bar{f}_z$ agree on $V$. $V$ is dense in its completion $\bar{V}$, and these functionals are all continuous, so they must agree on $\bar{V}$. Thus $x \mapsto \bar{f}_x$ is linear on $V$.

We thus know that $x \mapsto f_x$ is a linear transformation $V \to \bar{V}^*$ with bound $||f_x|| \leq ||x||$, so we may apply the BLT theorem again to extend it to $\bar{V} \to \bar{V}^*$.

This gives us $(x, y) = \bar{f}_x(y)$ and we must only check that it satisfies the correct properties. $f_x(y)$ is linear in $y$, so this is conjugate-linear in $y$; we just showed that $\bar{f}_x$ is linear in $x$. The same argument with the limits from part (1) gives the conjugate symmetry of $(\cdot, \cdot)$ because $f_x$ and $x \mapsto f_x$ are both continuous, and for $x, y \in V$, $f_x(y) = \bar{f}_y(x)$.

Finally, positive definiteness also follows from a similar argument with the limits.

**Problem 7 (R-S, II.6)** If $x, y \in \mathcal{M}^\perp$, $\alpha \in \mathbb{C}$, then for all $z \in M$, $(x, z) = (y, z) = 0$, so $(\alpha x, z) = (x + y, z) = 0$, so $\alpha x, x + y \in \mathcal{M}^\perp$.

If $x_n \to x, x_n \in \mathcal{M}^\perp$, then for all $z \in \mathcal{M}$, $(x_n, z) = 0$. $(x_n, z) \to (x, z)$, so $(x, z) = 0$ and $x \in \mathcal{M}^\perp$, so $\mathcal{M}^\perp$ is a closed linear subspace of $\mathcal{H}$.

Now, if $x_n \in \mathcal{M}, x_n \to x \in \mathcal{M}$, then if $z \in \mathcal{M}^\perp, (x_n, z) = 0$, so $(x, z) = 0$, so $x \in (\mathcal{M}^\perp)^\perp$, so we get one inclusion.

We also have that $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp = \bar{\mathcal{M}} \oplus \mathcal{M}^\perp$ and that $\mathcal{H} = \mathcal{M}^\perp \oplus \mathcal{M}^{\perp \perp}$. So, let $y \in \mathcal{M}^{\perp \perp}$. We may write $y$ uniquely as $y = z + w$ with $z \in \mathcal{M}, w \in \mathcal{M}^\perp$. The above inclusion tells us that $z \in \mathcal{M}^\perp$. We know that we may write $y$ uniquely as a sum of an element of $\mathcal{M}^{\perp \perp}$ and one of $\mathcal{M}^\perp$. One such expression is $y = y + 0$, so the uniqueness tells us that $z = y$ and $w = 0$. $z \in \mathcal{M}$, so this gives us the other inclusion.
Problem 8 (R-S, III.4) Let $||·||_0$ be the Euclidean norm (with respect to a basis $e_1, \ldots, e_n$) and let $||·||$ be any other norm on $\mathbb{R}^n$.

We claim that $x \mapsto ||x||$ is continuous with respect to the Euclidean norm. Write $x = c_1 e_1 + \ldots + c_n e_n$. Then by the triangle inequality,

$$||x|| \leq |c_1|||e_1|| + \ldots + |c_n|||e_n|| \leq (n \max ||e_i||) \max |c_i| \leq C ||x||_0,$$

where $C = n \max ||e_i||$, so if $||x - y||_0 \leq \frac{\epsilon}{C}$, then $||x - y|| \leq \epsilon$, so the map is continuous.

Now consider the region $K = \{ x : ||x||_0 = 1 \}$. This is just the unit sphere in $\mathbb{R}^n$ with the Euclidean norm, which is compact. $x \mapsto ||x||$ is continuous, so it attains a minimum $m$ and maximum $M$ on $K$. Note that $m > 0$ because $0 \notin K$. Thus, for $x \in K$, $m \leq ||x|| \leq M$. Now, for $x \in \mathbb{R}^n$, $x \neq 0$, $\frac{x}{||x||_0} \in K$, so

$$m \leq \frac{||x||}{||x||_0} \leq M,$$

i.e. $m ||x||_0 \leq ||x|| \leq M ||x||_0$,

so the two norms are equivalent.

Problem 9 (R-S, II.9) $f$ is a bounded linear functional on $\mathcal{M}$, so it extends to $\overline{\mathcal{M}}$ by the BLT theorem. The Riesz lemma then tells us that $f(x) = (x, z)$ for a unique $z \in \overline{\mathcal{M}}$. $\overline{\mathcal{M}} \subset \mathcal{H}$, so this extends to $\overline{f}(x) = (x, z)$ for all $x \in \mathcal{H}$. Now uniqueness: If $\overline{f}'$ is another such extension, then $\overline{f}'(x) = (x, z + y)$ for some $y \in \mathcal{H}$. For $x \in \overline{\mathcal{M}}$, $\overline{f}'(x) = (x, z)$, so $(x, y) = 0$ and $y \in \mathcal{M}^\perp$, so the bound for $\overline{f}'$ is $\sqrt{||z||^2 + ||y||^2} > ||z|| = C$ unless $y = 0$.

Problem 10 (R-S, II.5) Write $x = \sum_{n=1}^{N}(x_n, x)x_n + z$, and observe that for $n \leq N$,

$$(x_n, z) = (x_n, x) - \sum_{k=1}^{N}(x_k, x)(x_n, x_k) = (x_n, x) - (x_n, x) = 0,$$

so $z \perp x_n$ for all such $n$. Then we write $x - \sum_{n=1}^{N}c_n x_n = \sum_{n=1}^{N}(x_n, x) - c_n)x_n + z$, where $z \perp \sum_{n=1}^{N}(x_n, x) - c_n)x_n$, so

$$||x - \sum_{n=1}^{N}c_n x_n||^2 = ||z||^2 + \left|\sum_{n=1}^{N}(x_n, x) - c_n)x_n\right|^2 =

= ||z||^2 + \sum_{n=1}^{N}||(x_n, x) - c_n||^2,$$

which attains its minimum if $c_n = (x_n, x)$ for all $n$. 

\[5\]