MATH 256B: PROBLEM SET 4
DUE FRIDAY, MAY 26, 2017

Problem 1. (1) Show that the left and right microlocal parametrices are essentially unique, i.e. if $A \in \Psi^m_\infty(\mathbb{R}^n)$ is elliptic at α and B, B' are microlocal left parametrices, so

$$BA = I + E,\ B'A = I + E',\ \alpha \notin \mathrm{WF}'(E) \cup \mathrm{WF}'(E'),$$

then $\alpha \notin \mathrm{WF}'(B - B')$, and similarly for microlocal right parametrices.

(2) Show that if $\alpha \in \mathbb{R}^n \times \mathbb{S}^{n-1}$ and $\alpha \notin \mathrm{WF}^r(u)$ then there exists an open set U containing α such that for all $Q \in \Psi^0_\infty(\mathbb{R}^n)$ with $\mathrm{WF}'(Q) \subset U$, $Qu \in H^r$.

(3) Let $\pi : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}^n$ be the projection map. Show that $u \in \mathcal{S}'(\mathbb{R}^n)$ is locally H^s near a point $\bar{x}$ (i.e. there is $\phi \in C^\infty_c(\mathbb{R}^n)$ identically 1 near $\bar{x}$ such that $\phi u \in H^s(\mathbb{R}^n)$) if and only if $\pi^{-1}(\{\bar{x}\}) \cap \mathrm{WF}^s(u) = \emptyset$.

Problem 2. (1) Let $\xi = (\xi', \xi_n)$, $\xi' = (\xi_1, \dots, \xi_{n-1})$. Suppose $\phi \in \mathcal{S}(\mathbb{R}^{n-1})$ not identically 0 and $\chi \in C^\infty(\mathbb{R})$, with $\operatorname{supp}\chi \subset [1, \infty)$, $\chi(t) = 1$ for $t \geq 2$, and let $a(\xi) = \phi(\xi'/\sqrt{\xi_n})\chi(\xi_n)$. Let $u = \mathcal{F}^{-1}a$. Show that, writing $x = (x', x_n)$ as well, $\mathrm{WF}(u) = \{(0, 0, 0, \xi_n) :\ \xi_n > 0\}$, i.e. as a subset of $\mathbb{R}^n \times \mathbb{S}^{n-1}$, $\mathrm{WF}(u)$ is single point.

(2) Suppose that K is a compact subset of $\mathbb{R}^n \times \mathbb{S}^{n-1}$. Show that there exists a distribution u with $\mathrm{WF}(u) = K$.

Problem 3. A distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ is called H^s-conormal to an embedded codimension l submanifold Y of $\mathbb{R}^n$ if for any $k \geq 0$ and for any vector fields $V_1, \dots, V_k$ which are tangent to Y, $V_1 \dots V_k u \in H^s_{\mathrm{loc}}$.

(1) Show that if $U, \tilde{U}$ are open subsets of $\mathbb{R}^n$ containing a point $p \in Y$, resp. the origin, and F is a diffeomorphism $\tilde{U} \to U$ with $F(0) = p$ and $\tilde{Y} = \{x_{n-l+1} = \dots = x_n = 0\} \cap \tilde{U}$ with $F(\tilde{Y}) = Y \cap U$ then for u supported in U, u being H^s-conormal to Y is equivalent to F^*u being H^s-conormal to $\tilde{Y}$. Thus, for local theory one may as well assume that $Y = \{x_{n-l+1} = \dots = x_n = 0\}$.

(2) Let $D' = (D_{x_1}, \dots, D_{x_{n-l}})$, and for $\alpha \in \mathbb{N}^{n-l}$ let $(D')^\alpha = D^{\alpha_1}_{x_1} \dots D^{\alpha_{n-l}}_{x_{n-l}}$. Show that a compactly supported u is H^s-conormal to $Y = \{x_{n-l+1} = \dots = x_n = 0\}$ if and only if $(x_{i_1} D_{j_1}) \dots (x_{i_k} D_{j_k})(D')^\alpha \in H^s$ for all k, $i_1, \dots, i_k, j_1 \dots, j_k \in \{n - l + 1, \dots, n\}$ and α.

(3) Show that for a compactly supported u, being H^s-conormal to the origin is equivalent to the Fourier transform of u satisfying L^2 based versions of the symbol estimates of order s, i.e. for all $\beta \in \mathbb{N}^n$, $D^\beta \mathcal{F}u \in \langle\xi\rangle^{-s-|\beta|}L^2$.

(4) Show that if u is H^s-conormal to $Y = \{x_{n-l+1} = \dots = x_n = 0\}$, then for any r, $\mathrm{WF}(u), \mathrm{WF}^r(u) \subset N^*Y = \{(x', 0, 0, \xi'') :\ \xi'' \neq 0\}$ where $x' = (x_1, \dots, x_{n-l})$, $x'' = (x_{n-l+1}, \dots, x_n)$, and similarly with ξ.

Problem 4. Let

$$M_{0,\infty} = \mathbb{R}^n \times \mathbb{S}^{n-1},\ M_{\infty,0} = \mathbb{S}^{n-1} \times \mathbb{R}^n,\ M_{\infty,\infty} = \mathbb{S}^{n-1} \times \mathbb{S}^{n-1},$$
$$M = M_{0,\infty} \sqcup M_{\infty,0} \sqcup M_{\infty,\infty},$$

(disjoint union, so it has 3 disjoint pieces); this will be the locus of microlocalization in the scattering setting.

(1) Suppose $a \in S^m_\infty(\mathbb{R}^n; \mathbb{R}^n)$. Show first that for $(\bar{x}, \bar{\xi}) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ there exists a conic-in-the-second variable (i.e. $(x, \xi) \in \Gamma$ implies $(x, t\xi) \in \Gamma$ for $t > 0$) open Γ containing $(\bar{x}, \bar{\xi})$ on (i.e. restricted to) which a is elliptic, resp. in $S^{-\infty}_\infty(\mathbb{R}^n; \mathbb{R}^n)$, if and only if there exists $\lambda > 0$ and an open conic $\tilde{\Gamma}$ containing $(\bar{x}, \bar{\xi})$ such that on (i.e. restricted to) $(\bar{x}, \lambda\bar{\xi}) + \tilde{\Gamma}$, a is elliptic, resp. in $S^{-\infty}_\infty(\mathbb{R}^n; \mathbb{R}^n)$.

(2) For $A \in \Psi^{m,l}_{\mathrm{sc}}(\mathbb{R}^n)$, $A = q_L(a)$, $a \in S^m_{\mathrm{pr}}(\mathbb{R}^n; \mathbb{R}^n)$ a scattering symbol as on Problem Set 1, define

$$\mathrm{WF}'(A), \mathrm{ell}(A), \mathrm{Char}(A) \subset M$$

by considering sets of the type:

(a) $(\bar{x}, \bar{\xi}) \in M_{0,\infty}$: Γ open conic in the second variable containing $(\bar{x}, \bar{\xi})$, $\lambda > 0$, the set is $\Gamma_{0,\infty} = (\bar{x}, \lambda\bar{\xi}) + \Gamma$,

(b) $(\bar{x}, \bar{\xi}) \in M_{\infty,0}$: Γ open conic in the first variable (so $(x, \xi) \in \Gamma$ implies $(tx, \xi) \in \Gamma$ for $t > 0$) containing $(\bar{x}, \bar{\xi})$, $\lambda > 0$, the set is $\Gamma_{\infty,0} = (\lambda\bar{x}, \bar{\xi}) + \Gamma$,

(c) $(\bar{x}, \bar{\xi}) \in M_{\infty,\infty}$: Γ open conic in both the first and the second variable (i.e. $(x, \xi) \in \Gamma$ implies $(tx, t'\xi) \in \Gamma$ for all $t, t' > 0$) containing $(\bar{x}, \bar{\xi})$, $\lambda > 0$, the set is $\Gamma_{\infty,\infty} = \lambda(\bar{x}, \bar{\xi}) + \Gamma$,

and say, for $q \in M_{0,\infty}, M_{\infty,0}, M_{\infty,\infty}$ respectively, that $q \notin \mathrm{WF}'(A)$ if a set $\tilde{\Gamma} = \Gamma_{0,\infty}, \Gamma_{\infty,0}, \Gamma_{\infty,\infty}$, respectively, of the kind described above, exists such that $a|_{\tilde{\Gamma}} \in S^{-\infty,-\infty}_{\mathrm{pr}}(\mathbb{R}^n; \mathbb{R}^n)$. Similarly $q \in \mathrm{ell}(A)$ if $\tilde{\Gamma}$ as above exists such that there is $c > 0$ such that $|a|_{\tilde{\Gamma}}| \geq c\langle x\rangle^l\langle\xi\rangle^m$; $\mathrm{Char}(A) = \mathrm{ell}(A)^c$. Also, for $u \in \mathcal{S}'(\mathbb{R}^n)$ say that $q \in M$ is not in $\mathrm{WF}(u)$ if there exists $A \in \Psi^{0,0}_{\mathrm{sc}}(\mathbb{R}^n)$ which is elliptic at q and such that $Au \in \mathcal{S}(\mathbb{R}^n)$. (Often these are written as $\mathrm{WF}_{\mathrm{sc}}(u)$, etc., but here we drop the subscript.)

Show that in $M_{0,\infty} = \mathbb{R}^n \times \mathbb{S}^{n-1}$ these coincide with the usual definitions, and show that $\Psi_{\mathrm{sc}}(\mathbb{R}^n)$ is microlocal, i.e. $\mathrm{WF}(Au) \subset \mathrm{WF}'(A) \cap \mathrm{WF}(u)$, and finally that $\mathrm{WF}(u) = \emptyset$ implies $u \in \mathcal{S}(\mathbb{R}^n)$.

(3) Suppose $a \in S^m(\mathbb{R}^n)$ and $\phi \in S^1(\mathbb{R}^n)$ homogeneous degree 1 outside a compact subset (so there is $R > 0$ such that $\phi(tx) = t\phi(x)$ if $|x| > R$ and $t \geq 1$). Find $\mathrm{WF}(e^{i\phi}a)$. (Consider $\phi = 0$ first.)

(4) Let $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ be the Fourier transform. Show that for $A \in \Psi^{m,l}_{\mathrm{sc}}(\mathbb{R}^n)$, $\mathcal{F}A\mathcal{F}^{-1} \in \Psi^{l,m}_{\mathrm{sc}}(\mathbb{R}^n)$, and find the principal symbol of $\mathcal{F}A\mathcal{F}^{-1}$, and also $\mathrm{WF}'(\mathcal{F}A\mathcal{F}^{-1})$ in terms of those of A. Use these results to relate $\mathrm{WF}(u)$ and $\mathrm{WF}(\mathcal{F}u)$, $\mathrm{WF}(\mathcal{F}^{-1}u)$.