Problem 1. (1) Show that the left and right microlocal parametrices are essentially unique, i.e. if $A \in \Psi^{m}_{\infty}(\mathbb{R}^{n})$ is elliptic at $\alpha$ and $B, B'$ are microlocal left parametrices, so

$$BA = I + E, \quad B'A = I + E', \quad \alpha \notin WF'(E) \cup WF'(E'),$$

then $\alpha \notin WF'(B - B')$, and similarly for microlocal right parametrices.

(2) Show that if $\alpha \in \mathbb{R}^{n} \times S^{n-1}$ and $\alpha \notin WF'(u)$ then there exists an open set $U$ containing $\alpha$ such that for all $Q \in \Psi^{0}_{\infty}(\mathbb{R}^{n})$ with $WF'(Q) \subset U$, $Qu \in H'$.

(3) Let $\pi : \mathbb{R}^{n} \times S^{n-1} \rightarrow \mathbb{R}^{n}$ be the projection map. Show that $u \in S'(\mathbb{R}^{n})$ is locally $H^{s}$ near a point $\bar{x}$ (i.e. there is $\phi \in C_{c}^{\infty}(\mathbb{R}^{n})$ identically 1 near $\bar{x}$ such that $\phi u \in H^{s}(\mathbb{R}^{n})$) if and only if $\pi^{-1}(\{\bar{x}\}) \cap WF'(u) = \emptyset$.

Problem 2. (1) Let $\xi = (\xi^{'}, \xi_0)$, $\xi' = (\xi_1, \ldots, \xi_{n-1})$. Suppose $\phi \in S(\mathbb{R}^{n-1})$ not identically 0 and $\chi \in C_{c}^{\infty}(\mathbb{R})$, with supp $\chi \subset [1, \infty)$, $\chi(t) = 1$ for $t \geq 2$, and let $a(\xi) = \phi(\xi'/\sqrt{\xi_0}) \chi(\xi_0)$. Let $u = F^{-1}a$. Show that, writing $x = (x', x_n)$ as well, $WF(u) = \{(0, 0, 0, \xi_n) : \xi_n > 0\}$, i.e. as a subset of $\mathbb{R}^{n} \times S^{n-1}$, $WF(u)$ is single point.

(2) Suppose that $K$ is a compact subset of $\mathbb{R}^{n} \times S^{n-1}$. Show that there exists a distribution $u$ with $WF(u) = K$.

Problem 3. A distribution $u \in S'(\mathbb{R}^{n})$ is called $H^{s}$-conormal to an embedded codimension $l$ submanifold $Y$ of $\mathbb{R}^{n}$ if for any $k \geq 0$ and for any vector fields $V_1, \ldots, V_k$ which are tangent to $Y$, $V_1 \ldots V_k u \in H^{s}_{loc}$.

(1) Show that if $U, \tilde{U}$ are open subsets of $\mathbb{R}^{n}$ containing a point $p \in Y$, resp. the origin, and $F$ is a diffeomorphism $\tilde{U} \rightarrow U$ with $F(0) = p$ and $\tilde{Y} = \{x_{n-l+1} = \ldots = x_n = 0\} \cap \tilde{U}$ with $F(\tilde{Y}) = Y \cap U$ then for $u$ supported in $U$, $u$ being $H^{s}$-conormal to $Y$ is equivalent to $F^{*}u$ being $H^{s}$-conormal to $\tilde{Y}$. Thus, for local theory one may as well assume that $Y = \{x_{n-l+1} = \ldots = x_n = 0\}$.

(2) Let $D' = (D_{x_1}, \ldots, D_{x_n-l})$, and for $\alpha \in \mathbb{N}^{n-l}$ let $(D')^{\alpha} = D_{x_1}^{\alpha_1} \ldots D_{x_n-l}^{\alpha_{n-l}}$. Show that a compactly supported $u$ is $H^{s}$-conormal to $Y = \{x_{n-l+1} = \ldots = x_n = 0\}$ if and only if $(x_{1}, \ldots, x_{n-l}, D_{x_1} \ldots D_{x_n-l})(D')^{\alpha} \in H^{s}$ for all $k, 1, \ldots, j_{k}, j_{1}, \ldots, j_{k} \in \{n-l+1, \ldots, n\}$ and $\alpha$.

(3) Show that for a compactly supported $u$, being $H^{s}$-conormal to the origin is equivalent to the Fourier transform of $u$ satisfying $L^{2}$ based versions of the symbol estimates of order $s$, i.e. for all $\beta \in \mathbb{N}^{n}$, $D^{\beta}F_{\mu}(\xi) \in (\xi^{-s-|\beta|}L^{2})^{\infty}$.

(4) Show that if $u$ is $H^{s}$-conormal to $Y = \{x_{n-l+1} = \ldots = x_n = 0\}$, then for any $r$, $WF(u), WF^{r}(u) \subset N^{r}Y = \{(x', 0, 0, \xi'') : \xi'' \neq 0\}$ where $x' = (x_1, \ldots, x_{n-l})$, $x'' = (x_{n-l+1}, \ldots, x_n)$, and similarly with $\xi$. 

Problem 4. Let

$$M_{0, 0} = \mathbb{R}^{n} \times S^{n-1}, \quad M_{\infty, 0} = S^{n-1} \times \mathbb{R}^{n}, \quad M_{\infty, \infty} = S^{n-1} \times S^{n-1},$$

$$M = M_{0, \infty} \cup M_{\infty, 0} \cup M_{\infty, \infty},$$

(disjoint union, so it has 3 disjoint pieces); this will be the locus of microlocalization in the scattering setting.

(1) Suppose $a \in S^{m}_{sc}(\mathbb{R}^{n} ; \mathbb{R}^{n})$. Show first that for $(\bar{x}, \bar{\xi}) \in \mathbb{R}^{n} \times (\mathbb{R}^{n} \setminus \{0\})$ there exists a conic-in-the-second variable (i.e. $(x, \xi) \in \Gamma$ implies $(x, t\xi) \in \Gamma$ for $t > 0$) open $\Gamma$ containing $(\bar{x}, \bar{\xi})$ on (i.e. restricted to) which $a$ is elliptic, resp. in $S^{m}_{sc}(\mathbb{R}^{n}, \mathbb{R}^{n})$, if and only if there exists $\lambda > 0$ and an open conic $\tilde{\Gamma}$ containing $(\bar{x}, \bar{\xi})$ such that on (i.e. restricted to) $(\bar{x}, \lambda \bar{\xi}) + \tilde{\Gamma}$, $a$ is elliptic, resp. in $S^{m}_{sc}(\mathbb{R}^{n}, \mathbb{R}^{n})$.

(2) For $A \in \Psi^{m}_{sc}(\mathbb{R}^{n}), A = q_{L}(a), a \in S^{m}_{pr}(\mathbb{R}^{n}, \mathbb{R}^{n})$ a scattering symbol as on Problem Set 1, define

$$WF'(A), ell(A), Char(A) \subset M$$

by considering sets of the type:

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(a) \((\bar{x}, \bar{\xi}) \in M_{0,\infty}: \Gamma\) open conic in the second variable containing \((\bar{x}, \bar{\xi}), \lambda > 0\), the set is \(\Gamma_{0,\infty} = (\bar{x}, \lambda \bar{\xi}) + \Gamma\),

(b) \((\bar{x}, \bar{\xi}) \in M_{\infty,0}: \Gamma\) open conic in the first variable (so \((x, \xi) \in \Gamma\) implies \((tx, \xi) \in \Gamma\) for \(t > 0\)) containing \((\bar{x}, \bar{\xi}), \lambda > 0\), the set is \(\Gamma_{\infty,0} = \lambda (\bar{x}, \bar{\xi}) + \Gamma\),

(c) \((\bar{x}, \bar{\xi}) \in M_{\infty,\infty}: \Gamma\) open conic in both the first and the second variable (i.e. \((x, \xi) \in \Gamma\) implies \((tx, t'\xi) \in \Gamma\) for all \(t, t' > 0\)) containing \((\bar{x}, \bar{\xi}), \lambda > 0\), the set is \(\Gamma_{\infty,\infty} = \lambda (\bar{x}, \bar{\xi}) + \Gamma\),

and say, for \(q \in M_{0,\infty}, M_{\infty,0}, M_{\infty,\infty}\) respectively, that \(q \notin \text{WF}'(A)\) if a set \(\tilde{\Gamma} = \Gamma_{0,\infty}, \Gamma_{\infty,0}, \Gamma_{\infty,\infty}\), respectively, of the kind described above, exists such that \(a|_{\tilde{\Gamma}} \in S^{-\infty,-\infty}_{pr}(\mathbb{R}^n, \mathbb{R}^n)\). Similarly \(q \in \text{ell}(A)\) if \(\tilde{\Gamma}\) as above exists such that there is \(c > 0\) such that \(|a|_{\tilde{\Gamma}} \geq c (x)^l (\xi)^m; \text{Char}(A) = \text{ell}(A)^c\).

Also, for \(u \in S'(\mathbb{R}^n)\) say that \(q \in M\) is not in \(\text{WF}(u)\) if there exists \(A \in \Psi_{0,0}^{sc}(\mathbb{R}^n)\) which is elliptic at \(q\) and such that \(Au \in S(\mathbb{R}^n)\). (Often these are written as \(\text{WF}_{sc}(u)\), etc., but here we drop the subscript.)

Show that in \(M_{0,\infty} = \mathbb{R}^n \times S^{n-1}\) these coincide with the usual definitions, and show that \(\Psi_{sc}(\mathbb{R}^n)\) is microlocal, i.e. \(\text{WF}(Au) \subset \text{WF}'(A) \cap \text{WF}(u)\), and finally that \(\text{WF}(u) = \emptyset\) implies \(u \in S(\mathbb{R}^n)\).

(3) Suppose \(a \in S^m(\mathbb{R}^n)\) and \(\phi \in S^1(\mathbb{R}^n)\) homogeneous degree 1 outside a compact subset (so there is \(R > 0\) such that \(\phi(tx) = t\phi(x)\) if \(|x| > R\) and \(t \geq 1\)). Find \(\text{WF}(e^{i\phi}a)\). (Consider \(\phi = 0\) first.)

(4) Let \(\mathcal{F} : S(\mathbb{R}^n) \to S(\mathbb{R}^n)\) be the Fourier transform. Show that for \(A \in \Psi_{sc}^{m,1}(\mathbb{R}^n)\), \(\mathcal{F}A\mathcal{F}^{-1} \in \Psi_{sc}^{l,m}(\mathbb{R}^n)\), and find the principal symbol of \(\mathcal{F}A\mathcal{F}^{-1}\), and also \(\text{WF}'(\mathcal{F}A\mathcal{F}^{-1})\) in terms of those of \(A\). Use these results to relate \(\text{WF}(u)\) and \(\text{WF}(\mathcal{F}u)\), \(\text{WF}(\mathcal{F}^{-1}u)\).