MATH 256B: PROBLEM SET 3
DUE FRIDAY, MAY 5, 2017

Problem 1. (Precise elliptic regularity)

(1) Suppose $A \in \Psi^m_\infty(\mathbb{R}^n)$ is elliptic. Show that for all $\phi, \psi \in C^\infty_c(\mathbb{R}^n)$, with ψ equiv 1 on a neighborhood of $\operatorname{supp}\phi$, and for any s, N there exists C such that

$$\|\phi u\|_{H^s} \leq C(\|\psi A u\|_{H^{s-m}} + \|u\|_{H^{-N}})$$

whenever $u \in \mathcal{S}(\mathbb{R}^n)$, and furthermore just under the assumption $u \in H^{-N}(\mathbb{R}^n)$ with $\psi A u \in H^{s-m}(\mathbb{R}^n)$, one has $\phi u \in H^s(\mathbb{R}^n)$ and the estimate holds.

(2) If A is a differential operator, improve the conclusion by taking $\chi \in C^\infty_c(\mathbb{R}^n)$ identically 1 on a neighborhood of $\operatorname{supp}\psi$, and show that

$$\|\phi u\|_{H^s} \leq C(\|\psi A u\|_{H^{s-m}} + \|\chi u\|_{H^{-N}})$$

under the same assumptions as above.

Problem 2. (Elliptic regularity in the scattering algebra)

(1) Show in detail that if $A \in \Psi^{m,l}_{\mathrm{sc}}(\mathbb{R}^n)$, as on Problem Set 1, is elliptic, in the sense that $A = q_L(a)$ and there exist $c > 0$, $R > 0$ such that $|(x,\xi)| > R$ implies $|a(x,\xi)| > c\langle\xi\rangle^m\langle x\rangle^l$, then there exists a two-sided parametrix $B \in \Psi^{-m,-l}_{\mathrm{sc}}(\mathbb{R}^n)$ such that $AB - I, BA - I \in \Psi^{-\infty,-\infty}_{\mathrm{sc}}(\mathbb{R}^n)$.

(2) Show that if $A \in \Psi^{m,l}_{\mathrm{sc}}(\mathbb{R}^n)$ is elliptic, $u \in \mathcal{S}'(\mathbb{R}^n)$ and $Au \in H^{s-m,r-l} = \langle x\rangle^{-r}H^s$ (weighted Sobolev space), then $u \in H^{s,r}$. Show moreover that for $A \in \Psi^{m,l}_{\mathrm{sc}}(\mathbb{R}^n)$ elliptic, s, r, N, M real there exists $C > 0$ such that

$$\|u\|_{H^{s,r}} \leq C(\|Au\|_{H^{s-m,l-r}} + \|u\|_{H^{-N,-M}})$$

for $u \in H^{-N,-M}$ with $Au \in H^{s-m,r-l}$.

Problem 3. (Systems of pseudodifferential operators) Suppose $A \in \Psi^m_\infty(\mathbb{R}^n;\mathbb{C}^N)$ is an $N \times N$ matrix of pseudodifferential operators $A_{ij} \in \Psi^m_\infty(\mathbb{R}^n)$, $i, j = 1, \dots, N$, and define its principal symbol $\sigma_m(A)$ as the $N \times N$ matrix of principal symbols $(\sigma_m(A_{ij}))^N_{i,j=1} \in S^m(\mathbb{R}^n;\mathbb{R}^n;\mathcal{L}(\mathbb{C}^N))$. Let A act on Schwartz functions with values in $\mathbb{C}^N$, or equivalently N-tuples of Schwartz functions, $u = (u_j)^N_{j=1}$, by $(Au)_i = \sum_{j=1}^N A_{ij}u_j$.

(1) Show that for $A \in \Psi^m_\infty(\mathbb{R}^n;\mathbb{C}^N)$, $B \in \Psi^{m'}_\infty(\mathbb{R}^n;\mathbb{C}^N)$, we have $AB \in \Psi^{m+m'}(\mathbb{R}^n;\mathbb{C}^N)$, and the principal symbol of AB is the product (in the same order) of the principal symbols of A and B: $\sigma_{m+m'}(AB) = \sigma_m(A)\sigma_{m'}(B)$ (matrix multiplication).

(2) Show that $a = \sigma_m(A)$ is elliptic in the sense that there exists $b \in S^{-m}_\infty(\mathbb{R}^n;\mathbb{R}^n;\mathcal{L}(\mathbb{C}^N))$ such that $ab - 1 \in S^{-1}_\infty$ if and only if there exist $R, c > 0$ such that $|\det a(x,\xi)| \geq c\langle\xi\rangle^{mN}$ for $|\xi| > R$.

(3) Show that if $A \in \Psi^m_\infty(\mathbb{R}^n;\mathbb{C}^N)$ is elliptic then there exists $B \in \Psi^{-m}_\infty(\mathbb{R}^n;\mathbb{C}^N)$ such that $AB - I, BA - I \in \Psi^{-\infty}_\infty(\mathbb{R}^n;\mathbb{C}^N)$.

Problem 4. Let A be as in the previous problem.

(1) Show elliptic regularity: if A is elliptic, and $u \in H^{-M}(\mathbb{R}^n;\mathbb{C}^N)$ is an N-tuple of elements of $H^{-M}(\mathbb{R}^n)$ and $Au \in H^{s-m}(\mathbb{R}^n;\mathbb{C}^N)$ then $u \in H^s(\mathbb{R}^n;\mathbb{C}^N)$.

(2) Show that $A = \begin{pmatrix} a(x)(\partial_{x_1}g^{11}(x) + \partial_{x_2}g^{12}(x))a(x)^{-1} & -\partial_{x_2} \\ a(x)(\partial_{x_1}g^{21}(x) + \partial_{x_2}g^{22}(x))a(x)^{-1} & \partial_{x_1} \end{pmatrix}$ is elliptic if $a(x)$, resp. $g(x) = (g^{ij}(x))^2_{i,j=1}$ are a scalar real C^∞ function, resp. a symmetric 2×2 matrix of real C^∞ functions, bounded with all derivatives, acting as multiplication operators, and g is positive definite in the sense that there is $c > 0$ such that $\sum_{i,j=1}^2 g^{ij}(x)v_iv_j \geq c|v|^2$ for all $v \in \mathbb{R}^2$ and all $v \in \mathbb{R}^2$ while a is bounded below by a positive constant. (Note: interpreting the domain of A as consisting of 1-forms on $\mathbb{R}^2$, while the range is the pair consisting of 0-forms and 2-forms on $\mathbb{R}^2$, the second column of A is the exterior differential, and the first column is the divergence, relative to the (dual) Riemannian metric g, provided $a = \sqrt{\det g}$.)