Problem 1.  (Precise elliptic regularity)

(1) Suppose \( A \in \Psi_0^m(\mathbb{R}^n) \) is elliptic. Show that for all \( \phi, \psi \in C_0^\infty(\mathbb{R}^n) \), with \( \psi \) equiv 1 on a neighborhood of \( \text{supp} \, \phi \), and for any \( s, N \) there exists \( C \) such that
\[
\| \phi u \|_{H^s} \leq C(\| \psi Au \|_{H^{-m}} + \| u \|_{H^{-N}})
\]
whenever \( u \in S(\mathbb{R}^n) \), and furthermore just under the assumption \( u \in H^{-N}(\mathbb{R}^n) \) with \( \psi Au \in H^{s-m}(\mathbb{R}^n) \), one has \( \phi u \in H^s(\mathbb{R}^n) \) and the estimate holds.

(2) If \( A \) is a differential operator, improve the conclusion by taking \( \chi \in C_c^\infty(\mathbb{R}^n) \) identically 1 on a neighborhood of \( \text{supp} \, \psi \), and show that
\[
\| \phi u \|_{H^s} \leq C(\| \psi Au \|_{H^{-m}} + \| \chi u \|_{H^{-N}})
\]
under the same assumptions as above.

Problem 2.  (Elliptic regularity in the scattering algebra)

(1) Show in detail that if \( A \in \Psi_0^{m,j}(\mathbb{R}^n) \), as on Problem Set 1, is elliptic, in the sense that \( A = qL(\alpha) \) and there exist \( c > 0, R > 0 \) such that \( |a(x, \xi)| > R \) implies \( |a(x, \xi)| > c|\xi|^m(x) \), then there exists a two-sided parametrix \( B \in \Psi_{sc}^{m,-1}(\mathbb{R}^n) \) such that \( AB - I, BA - I \in \Psi_{sc}^{m,-\infty}(\mathbb{R}^n) \).

(2) Show that if \( A \in \Psi_0^{m,j}(\mathbb{R}^n) \) is elliptic, \( u \in \mathcal{S}(\mathbb{R}^n) \) and \( Au \in H^{s-m,r-1} \), then \( u \in H^{s+r} \). Show moreover that for \( A \in \Psi_0^{m,j}(\mathbb{R}^n) \) elliptic, \( s, r, N, M \) real there exists \( C > 0 \) such that
\[
\| u \|_{H^{s+r}} \leq C(\| Au \|_{H^{s-m,r-1}} + \| u \|_{H^{-N,-M}})
\]
for \( u \in H^{-N,-M} \) with \( Au \in H^{s-m,r-1} \).

Problem 3.  (Systems of pseudodifferential operators) Suppose \( A \in \Psi_0^{m}(\mathbb{R}^n; \mathbb{C}^N) \) is an \( N \times N \) matrix of pseudodifferential operators \( A_{ij} \in \Psi_0^{m}(\mathbb{R}^n) \), \( i, j = 1, \ldots, N \), and define its principal symbol \( \sigma_m(A) \) as the \( N \times N \) matrix of principal symbols \( \langle \sigma_m(A_{ij}) \rangle_{i,j=1}^N \in S^m(\mathbb{R}^n; \mathbb{R}^n; L(\mathbb{C}^N)) \). Let \( A \) act on Schwartz functions with values in \( \mathbb{C}^N \), or equivalently \( N \)-tuples of Schwartz functions, \( u = (u_j)_{j=1}^N \), by \( (Au)_i = \sum_{j=1}^N A_{ij}u_j \).

(1) Show that for \( A \in \Psi_0^{m}(\mathbb{R}^n; \mathbb{C}^N) \), \( B \in \Psi_0^{m'}(\mathbb{R}^n; \mathbb{C}^N) \), we have \( AB \in \Psi_0^{m+m'}(\mathbb{R}^n; \mathbb{C}^N) \), and the principal symbol of \( AB \) is the product (in the same order) of the principal symbols of \( A \) and \( B \): \( \sigma_{m+m'}(AB) = \sigma_m(A)\sigma_{m'}(B) \) (matrix multiplication).

(2) Show that \( a = \sigma_m(A) \) is elliptic in the sense that there exists \( b \in S^{-m}(\mathbb{R}^n; \mathbb{R}^n; L(\mathbb{C}^N)) \) such that \( ab - 1 \in S^{-\infty} \) if and only if there exist \( R, c > 0 \) such that \( |\det a(x, \xi)| \geq c|\xi|^m \) for \( |\xi| > R \).

(3) Show that if \( A \in \Psi_0^{m}(\mathbb{R}^n; \mathbb{C}^N) \) is elliptic then there exists \( B \in \Psi^{-m}(\mathbb{R}^n; \mathbb{C}^N) \) such that \( AB - I, BA - I \in \Psi^{-\infty}(\mathbb{R}^n; \mathbb{C}^N) \).

Problem 4.  Let \( A \) be as in the previous problem.

(1) Show elliptic regularity: if \( A \) is elliptic, and \( u \in H^{-M}(\mathbb{R}^n; \mathbb{C}^N) \) is an \( N \)-tuple of elements of \( H^{-M}(\mathbb{R}^n) \) and \( Au \in H^{s-m}(\mathbb{R}^n; \mathbb{C}^N) \) then \( u \in H^{s}(\mathbb{R}^n; \mathbb{C}^N) \).

(2) Show that \( A = \begin{pmatrix} a(x)(\partial_{x_1}g^{11}(x) + \partial_{x_2}g^{12}(x))a(x)^{-1} & -\partial_{x_2} \\ a(x)(\partial_{x_2}g^{21}(x) + \partial_{x_2}g^{22}(x))a(x)^{-1} & \partial_{x_1} \end{pmatrix} \) is elliptic if \( a(x) \), resp. \( g(x) = (g^{ij}(x))_{i,j=1}^2 \) are a scalar real \( C^\infty \) function, resp. a symmetric \( 2 \times 2 \) matrix of real \( C^\infty \) functions, bounded with all derivatives, acting as multiplication operators, and \( g \) is positive definite in the sense that there is \( c > 0 \) such that \( \sum_{i,j=1}^2 g^{ij}(x)u_iv_j \geq c|v|^2 \) for all \( v \in \mathbb{R}^2 \) and all \( v \in \mathbb{R}^2 \) while \( a \) is bounded below by a positive constant.  (Note: interpreting the domain of \( A \) as consisting of 1-forms on \( \mathbb{R}^2 \), while the range is the pair consisting of 0-forms and 2-forms on \( \mathbb{R}^2 \), the second column of \( A \) is the exterior differential, and the first column is the divergence, relative to the (dual) Riemannian metric \( g \), provided \( a = \sqrt{\det g} \).)