Problem 1. In class we showed that if $A \in \Psi^m(\mathbb{R}^n)$, $m < -n$, then the Schwartz kernel of $A$ (which a priori is a bounded continuous function given by a convergent integral) is $C^\infty$. Show carefully that the same statement is true for all $m$.

*Hint:* if $U, V$ are open sets with disjoint, compact, closure, then the Schwartz kernel of $A$ restricted to $U \times V$ is the same as that of $\phi A \psi$, where $\phi, \psi$ are $C^\infty$, $\equiv 1$ on $U$, resp. $V$, and have disjoint support. So show that $\phi A \psi$ is given by a $C^\infty$ Schwartz kernel by showing that for all $m'$ there exists $b$ such that $\phi A \psi = I(b)$, $b \in S^m(\mathbb{R}^{2n}; \mathbb{R}^n)$.

**Problem 2.**

1. Show that $A \in \Psi^m(\mathbb{R}^n)$ is ‘principally self-adjoint’, i.e. $A - A^* \in \Psi^{m-1}(\mathbb{R}^n)$, if and only if the principal symbol of $A$ is real, i.e. if and only if $A = q(a)$ and $\text{Im} a \in S^{m-1}(\mathbb{R}^n; \mathbb{R}^n)$.
2. Show that if $A \in \Psi^m(\mathbb{R}^n)$ has real principal symbol then there exists $B \in \Psi^{m-1}(\mathbb{R}^n)$ such that $\text{Im}(Au, u) = (Bu, u)$ for all $u \in S(\mathbb{R}^n)$, i.e. ‘$\langle Au, u \rangle$ is real to leading order’.
3. Show that if $a \in S^m(\mathbb{R}^n; \mathbb{R}^n)$ and $a$ is real modulo $S^{m-1}(\mathbb{R}^n; \mathbb{R}^n)$ then there is $R \in \Psi^{m-2}(\mathbb{R}^n)$ such that $\text{Re}(q_L(a)u, u) = \text{Re}(q_R(a)u, u)$ for all $u \in S(\mathbb{R}^n)$ (Thus, while $q_L(a)$ and $q_R(a)$ differ modulo $\Psi^{m-2}(\mathbb{R}^n)$, this particular inner product does not. One might say that if the principal symbol of $A$ is real then the real part of its symbol is well-defined modulo $S^{m-2}$.)

**Problem 3.**

1. For $a \in S^m(\mathbb{R}^n; \mathbb{R}^n), b \in S^m(\mathbb{R}^n; \mathbb{R}^n)$, find the formula for $c$, modulo $S^{m+m'-2}(\mathbb{R}^n; \mathbb{R}^n)$, such that $q_L(a)q_L(b) = q_L(c)$. (Melrose’s notes give the explicit formula in general, but compute this from scratch to get a feel for it.)
2. Show that if $a, b$ are real, then there is $R \in \Psi^{m+m'-2}(\mathbb{R}^n)$ such that $\text{Re}(q_L(a)q_L(b)u, u) = \text{Re}(q_L(ab)u, u) + \langle Ru, u \rangle$ for all $u \in S(\mathbb{R}^n)$.
3. Since the principal symbol of $q_L(a)q_L(b)$ is $ab$ (where $a, b$ not necessarily real), we have that the commutator of $q_L(a)$ and $q_L(b)$,

$$C = [q_L(a), q_L(b)] = q_L(a)q_L(b) - q_L(b)q_L(a) \in \Psi^{m+m'-1}(\mathbb{R}^n),$$

if $a \in S^m(\mathbb{R}^n; \mathbb{R}^n), b \in S^m(\mathbb{R}^n; \mathbb{R}^n)$. Show that $C = q_L(c)$ where

$$c = \frac{1}{i} \{a, b\} + c', c' \in S^{m+m'-2},$$

and where $\{a, b\}$ is the Poisson bracket of $a$ and $b$:

$$\{a, b\} = \sum_{j=1}^n (\partial_{x_j}a)(\partial_{x_j}b) - (\partial_{x_j}a)(\partial_{x_j}b),$$

and show that, modulo $S^{m+m'-2}$, $\{a, b\}$ is determined by $a$ modulo $S^{m-1}$ and $b$ modulo $S^{m'-1}$ (i.e. adding such terms to $a$, resp. $b$, leaves the Poisson bracket unchanged modulo $S^{m+m'-2}$). Thus, the principal symbol of $C$ is (the equivalence class of) $\frac{1}{i}\{a, b\}$, where $a, b$ may be replaced by any representatives of the respective principal symbols.

**Problem 4.** Let $\pi_W(x, y, \xi) = (\frac{x+y}{2}, \xi)$, and let $q_W = I \circ \pi_W$ be the Weyl quantization. Show that the Weyl quantization $q_W : S^m(\mathbb{R}^n; \mathbb{R}^n) \to \Psi^m(\mathbb{R}^n)$ is surjective, and given $a \in S^m(\mathbb{R}^{2n}; \mathbb{R}^n)$ find the asymptotic expansion of $a_W \in S^m(\mathbb{R}^n; \mathbb{R}^n)$ such that $I(a) = q_W(a_W)$.

*Hint:* write $a(z, w, \xi) = a(z + w, z - w, \xi)$, so $a(x, y, \xi) = a(\frac{x+y}{2}, \frac{x-y}{2}, \xi)$. Then expand $a$ in finite Taylor series around $w = 0$ (which, note, corresponds to the diagonal, $x = y$).

**Problem 5.**

1. For $a \in S^m(\mathbb{R}^n; \mathbb{R}^n), b \in S^m(\mathbb{R}^n; \mathbb{R}^n)$, find the formula for $c$, modulo $S^{m+m'-2}(\mathbb{R}^n; \mathbb{R}^n)$, such that $q_W(a)q_W(b) = q_W(c)$.
2. Give another proof of the result of Problem 3 regarding the principal symbol of the commutator of two pseudodifferential operators.