MATH 256B: PROBLEM SET 2
DUE FRIDAY, APRIL 28, 2017

Problem 1. In class we showed that if $A \in \Psi^m_\infty(\mathbb{R}^n)$, $m < -n$, then the Schwartz kernel of A (which a priori is a bounded continuous function given by a convergent integral) is C^∞. Show carefully that the same statement is true for all m.

Hint: if U, V are open sets with disjoint, compact, closure, then the Schwartz kernel of A restricted to $U \times V$ is the same as that of $\phi A \psi$, where ϕ, ψ are C^∞, $\equiv 1$ on U, resp. V, and have disjoint support. So show that $\phi A \psi$ is given by a C^∞ Schwartz kernel by showing that for all m' there exists b such that $\phi A \psi = I(b)$, $b \in S^{m'}_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$.

Problem 2.
(1) Show that $A \in \Psi^m_\infty(\mathbb{R}^n)$ is 'principally self-adjoint', i.e. $A - A^* \in \Psi^{m-1}_\infty(\mathbb{R}^n)$, if and only if the principal symbol of A is real, i.e. if and only if $A = q_L(a)$, and $\operatorname{Im} a \in S^{m-1}_\infty(\mathbb{R}^n; \mathbb{R}^n)$.
(2) Show that if $A \in \Psi^m_\infty(\mathbb{R}^n)$ has real principal symbol then there exists $B \in \Psi^{m-1}_\infty(\mathbb{R}^n)$ such that $\operatorname{Im}\langle Au, u\rangle = \langle Bu, u\rangle$ for all $u \in \mathcal{S}(\mathbb{R}^n)$, i.e. '$\langle Au, u\rangle$ is real to leading order'.
(3) Show that if $a \in S^m_\infty(\mathbb{R}^n; \mathbb{R}^n)$ and a is real modulo $S^{m-1}_\infty(\mathbb{R}^n; \mathbb{R}^n)$ then there is $R \in \Psi^{m-2}_\infty(\mathbb{R}^n)$ such that $\operatorname{Re}\langle q_L(a)u, u\rangle - \operatorname{Re}\langle q_R(a)u, u\rangle = \langle Ru, u\rangle$ for all $u \in \mathcal{S}(\mathbb{R}^n)$. (Thus, while $q_L(a)$ and $q_R(a)$ differ modulo $\Psi^{m-2}_\infty(\mathbb{R}^n)$, this particular inner product does not. One might say that if the principal symbol of A is real then the real part of its symbol is well-defined modulo S^{m-2}_∞.)

Problem 3.
(1) For $a \in S^m_\infty(\mathbb{R}^n; \mathbb{R}^n)$, $b \in S^{m'}_\infty(\mathbb{R}^n; \mathbb{R}^n)$, find the formula for c, modulo $S^{m+m'-2}_\infty(\mathbb{R}^n; \mathbb{R}^n)$, such that $q_L(a)q_L(b) = q_L(c)$. (Melrose's notes give the explicit formula in general, but compute this from scratch to get a feel for it.)
(2) Show that if a, b are real, then there is $R \in \Psi^{m+m'-2}_\infty(\mathbb{R}^n)$ such that $\operatorname{Re}\langle q_L(a)q_L(b)u, u\rangle = \operatorname{Re}\langle q_L(ab)u, u\rangle + \langle Ru, u\rangle$ for all $u \in \mathcal{S}(\mathbb{R}^n)$.
(3) Since the principal symbol of $q_L(a)q_L(b)$ is ab (where a, b not necessarily real), we have that the *commutator* of $q_L(a)$ and $q_L(b)$,
$$C = [q_L(a), q_L(b)] = q_L(a)q_L(b) - q_L(b)q_L(a) \in \Psi^{m+m'-1}_\infty(\mathbb{R}^n),$$
if $a \in S^m_\infty(\mathbb{R}^n; \mathbb{R}^n)$, $b \in S^{m'}_\infty(\mathbb{R}^n; \mathbb{R}^n)$. Show that $C = q_L(c)$ where
$$c = \frac{1}{i}\{a, b\} + c',\ c' \in S^{m+m'-2}_\infty,$$
and where $\{a, b\}$ is the *Poisson bracket* of a and b:
$$\{a, b\} = \sum_{j=1}^n ((\partial_{\xi_j} a)(\partial_{x_j} b) - (\partial_{x_j} a)(\partial_{\xi_j} b)),$$
and show that, modulo $S^{m+m'-2}_\infty$, $\{a, b\}$ is determined by a modulo S^{m-1}_∞ and b modulo $S^{m'-1}_\infty$ (i.e. adding such terms to a, resp. b, leaves the Poisson bracket unchanged modulo $S^{m+m'-2}_\infty$. Thus, the principal symbol of C is (the equivalence class of) $\frac{1}{i}\{a, b\}$, where a, b may be replaced by any representatives of the respective principal symbols.

Problem 4. Let $\pi_W(x, y, \xi) = (\frac{x+y}{2}, \xi)$, and let $q_W = I \circ \pi_W^*$ be the Weyl quantization. Show that the Weyl quantization $q_W : S^m_\infty(\mathbb{R}^n; \mathbb{R}^n) \to \Psi^m_\infty(\mathbb{R}^n)$ is surjective, and given $a \in S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$ find the asymptotic expansion of $a_W \in S^m_\infty(\mathbb{R}^n; \mathbb{R}^n)$ such that $I(a) = q_W(a_W)$.

Hint: write $\tilde{a}(z, w, \xi) = a(z + w, z - w, \xi)$, so $a(x, y, \xi) = \tilde{a}(\frac{x+y}{2}, \frac{x-y}{2}, \xi)$. Then expand $\tilde{a}$ in finite Taylor series around $w = 0$ (which, note, corresponds to the diagonal, $x = y$).

Problem 5.
(1) For $a \in S^m_\infty(\mathbb{R}^n; \mathbb{R}^n)$, $b \in S^{m'}_\infty(\mathbb{R}^n; \mathbb{R}^n)$, find the formula for c, modulo $S^{m+m'-2}_\infty(\mathbb{R}^n; \mathbb{R}^n)$, such that $q_W(a)q_W(b) = q_W(c)$.
(2) Give another proof of the result of Problem 3 regarding the principal symbol of the commutator of two pseudodifferential operators.