MATH 205B: PROBLEM SET 1
DUE FRIDAY, APRIL 21, 2017

Do Problems 2.2, 2.3 and 2.19 from Melrose's lecture notes (for 2.19 show that complex powers of positive elliptic symbols are symbols), as well as the following:

Problem 1 (Scattering pseudodifferential operators)**.** Let $S^{m,l}_{\mathrm{pr}}(\mathbb{R}^n;\mathbb{R}^n)$ be the set of C^∞ functions a on $\mathbb{R}^n\times\mathbb{R}^n$ such that

$$|(D_x^\alpha D_\xi^\beta a)(x,\xi)|\leq C_{\alpha\beta}\langle x\rangle^{l-|\alpha|}\langle\xi\rangle^{m-|\beta|}.$$

Let $\Psi^{m,l}_{\mathrm{sc}}(\mathbb{R}^n)$ be the image of $S^{m,l}_{\mathrm{pr}}(\mathbb{R}^n;\mathbb{R}^n)$ under the quantization map q_L. Show that if $A\in\Psi^{m,l}_{\mathrm{sc}}(\mathbb{R}^n)$ then there exists $b\in S^{m,l}_{\mathrm{pr}}(\mathbb{R}^n;\mathbb{R}^n)$ such that $A=q_R(b)$, and $a-b\in S^{m-1,l-1}_{\mathrm{pr}}(\mathbb{R}^n;\mathbb{R}^n)$. Show also that $A\in\Psi^{-\infty,-\infty}_{\mathrm{sc}}(\mathbb{R}^n)$ if and only if the Schwartz kernel K_A of A is in $\mathcal{S}(\mathbb{R}^{2n})$ (i.e. is a Schwartz function).

(Note: if $l=0$, as $a\in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$, the existence of $b\in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$ follows from the material covered in class. Thus, this problem strengthens what we obtained in class in this case.) (Hint: rather than proceeding from scratch, one can reduce to the $l=0$ case and show that the b produced in the $S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$ argument is in fact in $S^m_{\mathrm{pr}}(\mathbb{R}^n;\mathbb{R}^n)$ if $a\in S^m_{\mathrm{pr}}(\mathbb{R}^n;\mathbb{R}^n)$. When there are choices, make sure you make the 'right' ones to make your life simpler!)

Problem 2 (Algebra properties of Ψ_{sc})**.** Show that $\Psi^{\infty,\infty}_{\mathrm{sc}}(\mathbb{R}^n)$ is a bi-filtered $*$-algebra: $A\in\Psi^{m,l}_{\mathrm{sc}}(\mathbb{R}^n)$, $B\in\Psi^{m',l'}_{\mathrm{sc}}(\mathbb{R}^n)$ implies $AB\in\Psi^{m+m',l+l'}_{\mathrm{sc}}(\mathbb{R}^n)$, and $A^*\in\Psi^{m,l}_{\mathrm{sc}}(\mathbb{R}^n)$, with the principal symbol map a homomorphism of $*$-algebras:

$$\sigma_{\mathrm{sc},m,l}:\Psi^{m,l}_{\mathrm{sc}}(\mathbb{R}^n)\to S^{m,l}_{\mathrm{pr}}(\mathbb{R}^n,\mathbb{R}^n)/S^{m-1,l-1}_{\mathrm{pr}}(\mathbb{R}^n,\mathbb{R}^n),$$

and $\sigma_{\mathrm{sc},m+m',l+l'}(AB)=\sigma_{\mathrm{sc},m,l}(A)\sigma_{\mathrm{sc},m',l'}(B)$, $\sigma_{\mathrm{sc},m,l}(A^*)=\overline{\sigma_{\mathrm{sc},m,l}(A)}$.

Problem 3 (Borel lemma)**.** Suppose that $a_j\in C^\infty(\mathbb{T}^n_y)$ (complex valued). Show that there exists $a\in C^\infty(\mathbb{R}_x\times\mathbb{T}^n_y)$ such that $\partial_x^j a=a_j$. (Here the torus could be replaced by arbitrary compact manifolds, or $C^\infty(\mathbb{T}^n)$ could be replaced by $\mathcal{S}(\mathbb{R}^n)$ and one could ask for $a\in\mathcal{S}(\mathbb{R}^{n+1})$.)