This is an open book, notes, etc., exam. However, you must not discuss the problems with anyone except the instructor and the CA.

You may quote any theorem from the textbook, the lecture or the homework, provided you are not asked to prove it explicitly. You must cite any other reference precisely, and reproduce the argument in writing.

The total score is the number of points on Problems 1-3 plus the maximum of the scores on Problems 4 and 5.

Problem 1. (25 points)

1) Suppose that $X$ is a topological vector space (a (T1) topological space which is a vector space with the vector space operations continuous in the relevant (product) topologies), $T : \mathbb{F}^n \to X$ is linear, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, and $\mathbb{F}^n$ has the standard topology (given by any norm). Show that $T$ is continuous.

2) Suppose that $X$ is locally convex, $T : \mathbb{F}^n \to X$ is linear, $Y = \text{Ran} T$, and $T$ is injective. Show that $T : \mathbb{F}^n \to Y$ is a homeomorphism, i.e. $T^{-1} : Y \to \mathbb{F}^n$ is continuous. (Hint: let $S$ be the unit sphere in $\mathbb{F}^n$. What can you say about $T(S)$?)

3) Show that if $Y$ is a finite dimensional subspace of $X$, then $Y$ is closed. (Hint: show that if $p \in \bar{Y}$, then $p$ is in the closure of $T(K)$ for some $K \subset \mathbb{F}^n$ compact, where $T$ is an isomorphism as above.)

4) Show that if $Y$ is a closed subspace of $X$, and for each continuous seminorm $\rho$ on $X$, let $\rho_Y(x) = \inf\{\rho(x-y) : y \in Y\}$. Show that the $\rho_Y$ give rise to a locally convex topology on $X/Y$, and the resulting topology is the strongest topology in which the projection $\pi : X \to X/Y$ is continuous (i.e. it is the topology in which $A \subset X/Y$ is open if and only if $\pi^{-1}(A)$ is open).

Problem 2. (25 points) Let $s(\mathbb{Z})$ and $s'(\mathbb{Z})$ be as in Problem Set 7 (rapidly decreasing and polynomially growing sequences), $s'(\mathbb{Z})$ identified with $s(\mathbb{Z})^*$ via $j : s'(\mathbb{Z}) \to s(\mathbb{Z})^*$, $\iota : C^\infty(S^1) \to \mathcal{D}'(S^1)$ as in Problem Set 7.

1) Show that the Fourier series coefficient map

$$\mathcal{F} : \ell^2(S^1) \to \ell_2(\mathbb{Z}), \quad (\mathcal{F}f)_n = (e_n, f)_{L^2},$$

where $e_n(x) = (2\pi)^{-1/2} e^{inx}$, is a continuous linear map $\mathcal{F} : C^\infty(S^1) \to s(\mathbb{Z})$. (Hint: $\mathcal{F}\phi^{(k)} = (in)^k \mathcal{F}\phi$)

2) Show that the inverse map

$$\mathcal{F}^{-1} : \ell_2(\mathbb{Z}) \to L^2(S^1), \quad \mathcal{F}^{-1}(\{a_n\}) = \sum a_n e_n,$$

restricts to a continuous linear map $\mathcal{F}^{-1} : s(\mathbb{Z}) \to C^\infty(S^1)$.

3) Show that $\mathcal{F}$ extends to a continuous linear map $\mathcal{F} : \mathcal{D}'(S^1) \to s'(\mathbb{Z})$ as follows: for $\phi \in C^\infty(S^1)$, $\{a_n\} \in s(\mathbb{Z})$,

$$j(\mathcal{F}\phi)(\{a_n\}) = \iota(\phi)(\mathcal{F}^t\{a_n\}), \quad \mathcal{F}^t\{a_n\} = \sum a_n e_{-n},$$

and $\mathcal{F}^t : s(\mathbb{Z}) \to C^\infty(S^1)$ is continuous by (2). Now for $u \in \mathcal{D}'(S^1)$, define $\mathcal{F}u(\{a_n\}) = u(\mathcal{F}^t\{a_n\})$, and show that $\mathcal{F} : \mathcal{D}'(S^1) \to s'(\mathbb{Z})$ is continuous.
Similarly, it is easy to show that (but you do not need to write this down) $F^{-1}$ extends to a continuous linear map $F^{-1} : s'(\mathbb{Z}) \to D'(\mathbb{S}^1)$ by

$$F^{-1}((a_n)) = j((a_n))((F^{-1})^t\phi), \ a_n \in s'(\mathbb{Z}), \ \phi \in C^\infty(\mathbb{S}^1).$$

$$(F^{-1})^t\phi = \{e^{-n}\phi \} \ \text{for} \ \phi \in C^\infty(\mathbb{S}^1).$$

Show that $F F^{-1}$ is the identity on $s'(\mathbb{Z})$ and $F^{-1} F$ is the identity on $D'(\mathbb{S}^1)$.

4 Show that $F \frac{d}{dz} = (in) F$ on $D'(\mathbb{S}^1)$.

5 Show that if $u \in D'(\mathbb{S}^1)$ then there exist $g_1, g_2 \in C(\mathbb{S}^1)$ and $k > 0$ such that $u = \left(\frac{d}{dz}\right)^k g_1 + g_2$.

6 Show that $s(\mathbb{Z})$ is dense in $s'(\mathbb{Z})$, and use this to show that $C^\infty(\mathbb{S}^1)$ is dense $D'(\mathbb{S}^1)$.

**Problem 3. (25 points)**

1 Suppose that $X$ and $Y$ are Fréchet spaces over $\mathbb{C}$, and $B : X \times Y \to \mathbb{C}$ is jointly continuous and bilinear. Show that there exist continuous seminorms $\rho_X$ and $\rho_Y$ on $X$, resp. $Y$, such that $|B(x,y)| \leq \rho_X(x)\rho_Y(y)$ for all $x \in X, \ y \in Y$.

2 Let $A : s(\mathbb{Z}) \to s'(\mathbb{Z})$ be a continuous linear operator, where $s(\mathbb{Z})$ is the set of rapidly decreasing bi-infinite sequences with the standard Fréchet topology and $s'(\mathbb{Z}) = s(\mathbb{Z})^*$ is the set of polynomially growing bi-infinite sequences with the weak-* topology. For $n \in \mathbb{Z}$ let $e_n$ be the sequence whose $n$th entry is 1 and all other entries are 0. For $n, m \in \mathbb{Z}$ define $K_{nm}$ to be the $nm$th entry of $A e_n$.

Show that there exists $N > 0$ and $C > 0$ such that $|K_{nm}| \leq C(1 + |n| + |m|)^N$.

(Hint: Recall that the uniform boundedness principle, as well as some important consequences, hold in Fréchet spaces.)

3 Show that if $\kappa = (\kappa_{nm})$ is a bi-infinite matrix with $|\kappa_{nm}| \leq C(1 + |n| + |m|)^N$ for some $C > 0$ and $N > 0$ then there is a unique continuous linear map $B : s(\mathbb{Z}) \to s'(\mathbb{Z})$ such that $B e_n$ has $nm$th entry $\kappa_{nm}$.

4 Show the Schwartz kernel theorem: if $P : C^\infty(\mathbb{S}^1) \to D'(\mathbb{S}^1)$ is a continuous linear map then there is a unique $K \in D'(\mathbb{S}^1 \times \mathbb{S}^1)$ (the dual of $C^\infty(\mathbb{S}^1 \times \mathbb{S}^1)$) such that for $\phi, \psi \in C^\infty(\mathbb{S}^1)$, $(P \phi)(\psi) = K(\psi \otimes \phi)$ where $(\psi \otimes \phi)(x, y) = \psi(x) \phi(y)$.

You may use without proof (which would be easy in any case) the extensions of our results on the 1-dimensional Fourier series to the 2-dimensional one (i.e. the one on $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$).

**Problem 4. (25 points; do at least one of Problems 4 and 5)** Let $S(\mathbb{R}^n)$ denote the space of Schwartz functions on $\mathbb{R}^n$, and let $\partial_j$ be the partial derivative in the $j$th coordinate. Define the Fourier transform $F : L^1(\mathbb{R}^n) \to C^\infty_0(\mathbb{R}^n)$ by $(F f)(\xi) = \int e^{i\xi \cdot x} f(x) dx$, and define the `inverse Fourier transform' $F^{-1} : L^1(\mathbb{R}^n) \to C^\infty_0(\mathbb{R}^n)$ by $(F^{-1} f)(x) = (2\pi)^{-n} \int e^{ix \cdot \xi} f(\xi) d\xi$.

1 Show that $F, F^{-1}$ restrict to continuous linear maps $F, F^{-1} : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$. (Hint: show that $F \partial_j f = i\xi_j F f$, $F(x_j f) = i\partial_j F f$.)

2 Show that $F^{-1} F : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ commutes with $x_j$ and $\partial_j$, and similarly for $F F^{-1}$.

3 Show that if $T$ is a linear map on $S(\mathbb{R}^n)$ that commutes with $x_j$ and $\partial_j$ for all $j$, then $T$ is a multiple of the identity, i.e. there exists $c \in \mathbb{C}$ such that $T \phi = c \phi$ for all $\phi \in S(\mathbb{R}^n)$.

4 Show that $F^{-1}$ is indeed the inverse of $F$ on $S(\mathbb{R}^n)$ by computing this constant. (Hint: one convenient computation is to find $F \phi$ where $\phi(x) = e^{-a|x|^2}$, $a > 0$.)

5 Define $F : S'(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ and $F^{-1} : S'(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ so that these maps are both continuous and restrict to the already defined maps on $S(\mathbb{R}^n)$, and show that they are still the inverses of each other, and satisfy $F \partial_j f = i\xi_j F f$, $F(x_j f) = i\partial_j F f$, $f \in S'(\mathbb{R}^n)$.

6 If $P$ is a polynomial, $P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$, let $P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$, where $D_j = -i\partial_j$.

We say that $P(D)$ is elliptic if $p(\xi) = \sum_{|\alpha| = m} a_\alpha \xi^\alpha$ satisfies $\xi \neq 0 \Rightarrow p(\xi) \neq 0$. 

Show that if $P(D)$ is elliptic and $P(\xi)$ does not vanish for any $\xi \in \mathbb{R}^n$ then $P(D) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ and $P(D) : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ are invertible, and find an expression for the inverse.

(7) If $P$ is elliptic, $u \in \mathcal{S}'(\mathbb{R}^n)$ and $P(D)u \in \mathcal{S}(\mathbb{R}^n)$, show that $u \in C^\infty(\mathbb{R}^n)$. (This is a weak version of elliptic regularity. A stronger version would say that if $P(D)u$ is $C^\infty$ near some $x_0 \in \mathbb{R}^n$, then so is $u$. There are also versions in terms of Sobolev spaces.)

**Problem 5.** (25 points; do at least one of Problems 4 and 5)

Suppose that $f$ is a real valued $C^\infty$ function on $\mathbb{R}^n$ and $K \subset \mathbb{R}^n$ is compact, $u \in C^\infty(\mathbb{R}^n)$ supported in $K$, and consider

$$I(\omega) = \int e^{i\omega f(x)} u(x) \, dx, \quad \omega \geq 1.$$ 

(1) If $df$ never vanishes (i.e. $f$ has no critical points), show that for all $N$, $|I(\omega)| \leq C_N \omega^{-N}$, i.e. $I(\omega)$ decreases rapidly as $\omega \to \infty$. (Hint: use a partition of unity to reduce to the case $\partial_j f \neq 0$ for some $j$, and then integrate by parts.)

(2) Suppose $df$ has a non-degenerate zero at 0, i.e. $|df(x)| \geq c|x|$, $c > 0$, and no other zeros. Show that if $u \in C^K$, with $D^\alpha u$ vanishing to order $2k - |\alpha|$ at 0 for $|\alpha| \leq k$ in the sense that $|D^\alpha u| \leq C|x|^{2k - |\alpha|}$, then $|I(\omega)| \leq C_k \omega^{-k}$. (Thus, for a $C^\infty$ function $u$, given any $k$, only finitely many terms in the Taylor expansion of $u$ around 0 can matter for the behavior, modulo $C_k \omega^{-k}$, of $I(\omega)$ as $\omega \to \infty$.)

(3) Show that the inverse Fourier transform of $v(x) = e^{i(Ax,x)/2}$, where $A$ is a symmetric invertible linear operator on $\mathbb{R}^n$, is $(2\pi)^{-n/2} |\det A|^{-1/2} e^{i(\pi/4) \text{sgn} A} e^{-i(A^{-1}x,x)/2}$, where $\text{sgn} A$ is the difference of the number of positive and negative eigenvalues of $A$. (Hint: reduce to the $n = 1$ case, when $A$ can be thought of as a real number, and consider the more general case $\text{Re} A \geq 0$, taking a limit from $\text{Re} A > 0$.)

(4) Suppose that $f(x) = f(0) + \frac{1}{2} \langle Ax, x \rangle + R(x)$, where $R$ vanishes cubically at 0, and $df$ only has a zero at 0. Let

$$I(\omega, s) = \int e^{i\omega f(x)} u(x) \, dx, \quad f_s(x) = f(0) + \frac{1}{2} \langle Ax, x \rangle + sR(x), s \in [0,1].$$

With $I^{(k)}$ denoting the $k$th derivative in $s$, show that $|I^{(2k)}(\omega, s)| \leq C_k \omega^{-k}$, and use this (with Taylor’s theorem) to show that $|I(\omega) - \sum_{j=0}^{2k-1} \frac{1}{j!} I^{(j)}(\omega, 0)| \leq C_k \omega^{-k}$.

(5) Prove the stationary phase lemma: there exist constants $a_j, C_k$ such that

$$|I(\omega) - e^{i\omega f(0)} \sum_{j=0}^{k-1} a_j \omega^{-j-n/2}| \leq C_k \omega^{-k-n/2},$$

and find $a_0$. (Hint: use the previous part to reduce to evaluating an integral with purely quadratic phase function, and use that only finitely many terms in the Taylor expansion of $u$ may matter for this result, so one can replace $u$ by a polynomial times a cutoff function, identically 1 near 0, and the choice of the latter does not matter.)

(6) Find the asymptotic expansion near infinity of the inverse Fourier transform of the delta distribution $\delta_{S^{n-1}}$ on the unit sphere $S^{n-1} \subset \mathbb{R}^n$; here $\delta_{S^{n-1}}(\phi) = \int_{S^{n-1}} \phi$, with the integral taken with respect to the standard spherical measure.