This is a closed book, closed notes, no computers, etc., exam.
There are 4 problems. Solve all of them.

**Problem 1.** Suppose that \((X, \| \cdot \|_X)\) is a Banach space and \(Y\) is a closed linear subspace of \(X\). Suppose that \(\| \cdot \|_Y\) is a norm on \(Y\), \((Y, \| \cdot \|_Y)\) is a Banach space, and the inclusion map \(\iota : Y \to X\) (with \(\iota(y) = y\)) is continuous from \((Y, \| \cdot \|_Y)\) to \((X, \| \cdot \|_X)\). Show that the restriction of the norm \(\| \cdot \|_X\) to \(Y\) is equivalent to \(\| \cdot \|_Y\), i.e. there exists a constant \(C > 0\) such that \(\|y\|_X \leq C\|y\|_Y\) and \(\|y\|_Y \leq C\|y\|_X\) for all \(y \in Y\).

**Solution.** As \(Y\) is a closed subspace of \(X\), \((Y, \| \cdot \|_X)\) is a Banach space. (Every Cauchy sequence in \(Y\) has a limit in \(X\) as \(X\) is complete, but \(Y\) is closed, so the limit must lie in \(Y\).) The continuity of \(\iota\) means that \(\iota\) is bounded, i.e. there exists \(C_1 > 0\) such that for \(y \in Y\), \(\|y\|_X = \|\iota(y)\|_X \leq C_1\|y\|_Y\). Now \(\iota\) is a bijection from \((Y, \| \cdot \|_Y)\) to \((Y, \| \cdot \|_X)\), is continuous, so by the inverse function theorem, its inverse is also continuous, hence bounded. That is, there exists \(C_2 > 0\) such that \(\|y\|_Y = \|\iota^{-1}(y)\|_Y \leq C_2\|y\|_X\), completing the proof.

**Problem 2.** Let \(\ell_{\infty, \mathbb{R}}\) be the Banach space of bounded real valued sequences with the norm \(\|\{a_n\}\|_{\ell_{\infty, \mathbb{R}}} = \sup\{|a_n| : n \in \mathbb{N}\}\). Show that there is an element \(\lambda\) of \(\ell_{\infty, \mathbb{R}}^*\) such that for all sequences \(\{a_n\} \in \ell_{\infty, \mathbb{R}}\),

\[
\liminf_{n \to \infty} a_n \leq \lambda(\{a_n\}) \leq \limsup_{n \to \infty} a_n.
\]

(Hint: first ignore the inequality involving \(\liminf_{n \to \infty} a_n\).)

**Solution.** Let \(p : \ell_{\infty, \mathbb{R}} \to \mathbb{R}\) be defined by \(p(\{a_n\}) = \limsup_{n \to \infty} a_n\). Then \(p\) is convex, since for \(t \in [0, 1]\), \(\{a_n\}, \{b_n\} \in \ell_{\infty, \mathbb{R}}\),

\[
p(t\{a_n\} + (1-t)\{b_n\}) = p(t\{ta_n + (1-t)b_n\}) \leq \limsup_{n \to \infty} ta_n + \limsup_{n \to \infty} (1-t)b_n = tp(\{a_n\}) + (1-t)p(\{b_n\}),
\]

where we used that \(t, 1-t \geq 0\). Let \(\lambda_0\) be the zero functional on \(\{0\}\) (the space consisting of the zero sequence), note that \(\lambda_0(0) = 0 = \limsup_{n \to \infty} 0\). By the Hahn-Banach theorem, there is a linear extension \(\lambda\) of \(\lambda_0\) to \(\ell_{\infty, \mathbb{R}}\) such that \(\lambda(\{a_n\}) \leq p(\{a_n\}) = \limsup_{n \to \infty} a_n\) for all \(\{a_n\}\). Applying this to \(\{-a_n\}\),

\[
-\lambda(\{a_n\}) = \lambda(\{-a_n\}) \leq \limsup_{n \to \infty} -a_n = -\liminf_{n \to \infty} a_n,
\]

so taking the negative of both sides gives

\[
\liminf_{n \to \infty} a_n \leq \lambda(\{a_n\}) \leq \limsup_{n \to \infty} a_n.
\]

Note that \(\lambda\) is thus bounded, for

\[
\limsup_{n \to \infty} a_n \leq \sup_{n} a_n \leq \sup |a_n|, \quad \liminf_{n \to \infty} a_n \geq \inf_{n} a_n \geq -\sup_{n} |a_n|,
\]

so \(\lambda(\{a_n\}) \leq \sup_{n} |a_n| = \|\{a_n\}\|_{\ell_{\infty, \mathbb{R}}}\).
Problem 3. Suppose that \((X, \|\cdot\|)\) is a Banach space and \(Y\) is a closed subspace. Let \(Y^\perp\) denote the annihilator of \(Y\) in \(X^*\), i.e. \(Y^\perp = \{f \in X^* : f|_Y = 0\}\).

1. Show that \(Y^\perp\) is closed in \(X^*\).
2. If \(f \in X^*\), let \([f]\) denote the image of \(f\) in \(X^*/Y^\perp\). Let \(j : X^*/Y^\perp \to Y^*\) be defined by \(j([f]) = f|_Y\). Show that \(j\) is well-defined, and is an isometric isomorphism from \(X^*/Y^\perp\) onto \(Y^*\).

Solution. If \(\{f_n\}\) is a sequence in \(Y^\perp \subset X^*\), \(f_n \to f \in X^*\), then for \(y \in Y\), \(f_n(y) = 0\), so \(f(y) = \lim f_n(y) = 0\) as the map \(X^* \ni g \mapsto g(y)\) is continuous. Thus, \(Y^\perp\) is closed.

\(j\) is well-defined because if \([f] = [g]\) then \(f - g \in Y^\perp\) hence \((f - g)(y) = 0\) for \(y \in Y\), i.e. \((f - g)|_Y = 0\). It is then immediately linear.

If \(j([f]) = 0\) then \(f|_Y = 0\), so \(f(y) = 0\) for all \(y \in Y\), so \(f \in Y^\perp\), hence \([f] = 0\), showing that \(j\) is injective.

If \(h \in Y^*\), so in particular \(|h(y)| \leq C\|y\|\) for \(y \in Y\), \(C = \|h\|_{Y^*}\), by Hahn-Banach there exists an extension \(f\) of \(h\) to \(X\) such that \(|f(x)| \leq C\|x\|\) for all \(x \in X\), i.e. \(f \in X^*\). Then \(j([f]) = f|_Y = h\), so \(j\) is surjective.

If \(f \in X^*\), then \(\|\,[f]\|_{X^*/Y^\perp} = \inf_{g \in Y^\perp} \|f - g\|_{X^*}\). In particular, for \(\epsilon > 0\), there is \(g \in Y^\perp\) such that \(|f - g|_{X^*} \leq \|f\|_{X^*/Y^\perp} + \epsilon\). Then for \(y \in Y\), \(|f(y)| - |g(y)| \leq |f - g|_{X^*} \|y\|_{Y^*}\), so \(\|j([f])\|_{Y^*} \leq \|f - g\|_{X^*} \|y\|_{Y^*} \leq \|f\|_{X^*/Y^\perp} + \epsilon\). As \(\epsilon > 0\) is arbitrary, \(\|j([f])\|_{Y^*} \leq \|f\|_{X^*/Y^\perp}\).

On the other hand, if \(h \in Y^*\), we can extend \(h\) to some \(\phi \in X^*\) with \(|\phi|_{X^*} = \|h\|_{Y^*}\), as above. So if \(f \in X^*\), \(h = j([f])\), then \(\phi \in X^*\) satisfies \(\phi|_Y = f\), so \(f - \phi \in Y^\perp\). Thus, \(\|j([f])\|_{Y^*} = \|f - \phi\|_{X^*} = \|\phi\|_{X^*} = \|\phi|_Y\|_{Y^*}\).

Combining these gives \(\|f\|_{X^*/Y^\perp} = \|j([f])\|_{Y^*}\), so \(j\) is indeed an isometric isomorphism.

Problem 4. Suppose that \(X\) is a separable Hilbert space with inner product \((.,.)_X\), and \(Y\) is a linear subspace. Suppose \(Y\) is a separable Hilbert space with respect to an inner product \((.,.)_Y\), and the inclusion map \(i : Y \to X\) is continuous.

Let \(\{x_k\}\) be a sequence in \(Y\) such that there exists \(C > 0\) with \(\|x_k\|_Y \leq C\) for all \(k\), and such that \(\{x_k\}\) converges to some \(x \in X\) (in the topology of \(X\)). Show that \(x \in Y\).

You may use that closed balls in a separable Hilbert space are sequentially compact in the topology of weak convergence, i.e. every bounded sequence has a weakly convergent subsequence.

Solution. As \(\{x_k\}\) lies in the closed ball of radius \(C\) around \(0\) in \(Y\), which is weakly sequentially compact, \(\{x_k\}\) has a \(Y\)-weakly convergent subsequence, \(\{x_{n(k)}\}\); let \(y \in Y\) be the limit. Thus, for all \(z \in Y\), \(\lim_{k \to \infty} (x_{n(k)}, z)_Y = (y, z)_Y\). As \(i : Y \to X\) is continuous (with respect to the respective norms), for \(w \in X\),

\[(x_{n(k)}, w)_X = (i(x_{n(k)}), w)_X = (x_{n(k)}, i^*w)_Y,\]

where \(i^*\) is the adjoint of \(i\), so \(\lim_{k \to \infty} (x_{n(k)}, w)_X = (y, i^*w)_Y = (ty, w)_X\), so \(x_{n(k)} \to y\) \(X\)-weakly as well. But \(\{x_{n(k)}\}\) converges in the \(X\)-norm topology to \(x\) (i.e. \(\|x_{n(k)} - x\|_X \to 0\) as \(k \to \infty\)), hence \(X\)-weakly, i.e. \((x_{n(k)}, w)_X \to (x, w)_X\), so \((x, w)_X = (y, w)_X\) for all \(w \in X\), and thus \(x = y\) (as \((x - y, x - y)_X = 0\) if we take \(w = x - y \in X\)). This shows that \(x \in Y\).