MATH 205B: PRACTICE MIDTERM

This is a closed book, closed notes, no computers, etc., exam.
There are 4 problems. Solve all of them.

Problem 1. *Suppose that $(X, \|.\|_X)$ is a Banach space and Y is a closed linear subspace of X. Suppose that $\|.\|_Y$ is a norm on Y, $(Y, \|.\|_Y)$ is a Banach space, and the inclusion map $\iota : Y \to X$ (with $\iota(y) = y$) is continuous from $(Y, \|.\|_Y)$ to $(X, \|.\|_X)$. Show that the restriction of the norm $\|.\|_X$ to Y is equivalent to $\|.\|_Y$, i.e. there exists a constant $C > 0$ such that $\|y\|_X \leq C\|y\|_Y$ and $\|y\|_Y \leq C\|y\|_X$ for all $y \in Y$.*

Solution. As Y is a closed subspace of X, $(Y, \|.\|_X)$ is a Banach space. (Every Cauchy sequence in Y has a limit in X as X is complete, but Y is closed, so the limit must lie in Y.) The continuity of ι means that ι is bounded, i.e. there exists $C_1 > 0$ such that for $y \in Y$, $\|y\|_X = \|\iota(y)\|_X \leq C_1\|y\|_Y$. Now ι is a bijection from $(Y, \|.\|_Y)$ to $(Y, \|.\|_X)$, is continuous, so by the inverse function theorem, its inverse is also continuous, hence bounded. That is, there exists $C_2 > 0$ such that $\|y\|_Y = \|\iota^{-1}(y)\|_Y \leq C_2\|y\|_X$, completing the proof.

Problem 2. *Let $\ell_{\infty,\mathbb{R}}$ be the Banach space of bounded real valued sequences with the norm $\|\{a_n\}\|_{\ell_{\infty,\mathbb{R}}} = \sup\{|a_n| : n \in \mathbb{N}\}$. Show that there is an element λ of $\ell^*_{\infty,\mathbb{R}}$ such that for all sequences $\{a_n\} \in \ell_{\infty,\mathbb{R}}$,*

$$\liminf_{n\to\infty} a_n \leq \lambda(\{a_n\}) \leq \limsup_{n\to\infty} a_n.$$

(Hint: first ignore the inequality involving $\liminf_{n\to\infty} a_n$.)

Solution. Let $p : \ell_{\infty,\mathbb{R}} \to \mathbb{R}$ be defined by $p(\{a_n\}) = \limsup_{n\to\infty} a_n$. Then p is convex, since for $t \in [0, 1]$, $\{a_n\}, \{b_n\} \in \ell_{\infty,\mathbb{R}}$,

$$\begin{aligned} p(t\{a_n\} + (1-t)\{b_n\}) &= p(\{ta_n + (1-t)b_n\}) = \limsup_{n\to\infty}(ta_n + (1-t)b_n) \\ &\leq \limsup_{n\to\infty} ta_n + \limsup_{n\to\infty}(1-t)b_n = tp(\{a_n\}) + (1-t)p(\{b_n\}), \end{aligned}$$

where we used that $t, 1-t \geq 0$. Let λ_0 be the zero functional on $\{0\}$ (the space consisting of the zero sequence), note that $\lambda_0(0) = 0 = \limsup_{n\to\infty} 0$. By the Hahn-Banach theorem, there is a linear extension λ of λ to $\ell_{\infty,\mathbb{R}}$ such that $\lambda(\{a_n\}) \leq p(\{a_n\}) = \limsup_{n\to\infty} a_n$ for all $\{a_n\}$. Applying this to $\{-a_n\}$,

$$-\lambda(\{a_n\}) = \lambda(\{-a_n\}) \leq \limsup_{n\to\infty} -a_n = -\liminf_{n\to\infty} a_n,$$

so taking the negative of both sides gives

$$\liminf_{n\to\infty} a_n \leq \lambda(\{a_n\}) \leq \limsup_{n\to\infty} a_n.$$

Note that λ is thus bounded, for

$$\limsup_{n\to\infty} a_n \leq \sup_n a_n \leq \sup_n |a_n|, \ \liminf_{n\to\infty} a_n \geq \inf_n a_n \geq -\sup_n |a_n|,$$

so $|\lambda(\{a_n\})| \leq \sup_n |a_n| = \|\{a_n\}\|_{\ell_{\infty,\mathbb{R}}}$.

Problem 3. *Suppose that $(X, \|.\|)$ is a Banach space and Y is a closed subspace. Let $Y^\perp$ denote the annihilator of Y in X^*, i.e. $Y^\perp = \{f \in X^* : \ f|_Y = 0\}$.*

(1) *Show that $Y^\perp$ is closed in X^*.*
(2) *If $f \in X^*$, let $[f]$ denote the image of f in $X^*/Y^\perp$. Let $j : X^*/Y^\perp \to Y^*$ be defined by $j([f]) = f|_Y$. Show that j is well-defined, and is an isometric isomorphism from $X^*/Y^\perp$ onto Y^*.*

Solution. If $\{f_n\}$ is a sequence in $Y^\perp \subset X^*$, $f_n \to f \in X^*$, then for $y \in Y$, $f_n(y) = 0$, so $f(y) = (\lim f_n)(y) = \lim f_n(y) = 0$ as the map $X^* \ni g \mapsto g(y)$ is continuous. Thus, $Y^\perp$ is closed.

j is well-defined because if $[f] = [g]$ then $f - g \in Y^\perp$ hence $(f - g)(y) = 0$ for $y \in Y$, i.e. $(f - g)|_Y = 0$, i.e. $f|_Y = g|_Y$. It is then immediately linear.

If $j([f]) = 0$ then $f|_Y = 0$, so $f(y) = 0$ for all $y \in Y$, so $f \in Y^\perp$, hence $[f] = [0]$, showing that j is injective.

If $h \in Y^*$, so in particular $|h(y)| \leq C\|y\|$ for $y \in Y$, $C = \|h\|_{Y^*}$, by Hahn-Banach there exists an extension f of h to X such that $|f(x)| \leq C\|x\|$ for all $x \in X$, i.e. $f \in X^*$. Then $j([f]) = f|_Y = h$, so f is surjective.

If $f \in X^*$, then $\|[f]\|_{X^*/Y^\perp} = \inf_{g \in Y^\perp} \|f - g\|_{X^*}$. In particular, for $\epsilon > 0$, there is $g \in Y^\perp$ such that $\|f - g\|_{X^*} \leq \|[f]\|_{X^*/Y^\perp} + \epsilon$. Then for $y \in Y$, $|f(y)| = |(f - g)(y)| \leq \|f - g\|_{X^*}\|y\|_Y$, so $\|j([f])\|_{Y^*} \leq \|f - g\|_{X^*} \leq \|[f]\|_{X^*/Y^\perp} + \epsilon$. As $\epsilon > 0$ is arbitrary, $\|j([f])\|_{Y^*} \leq \|[f]\|_{X^*/Y^\perp}$.

On the other hand, if $h \in Y^*$, we can extend h to some $\phi \in X^*$ with $\|\phi\|_{X^*} = \|h\|_{Y^*}$ as above. So if $f \in X^*$, $h = j([f])$, then $\phi \in X^*$ satisfies $\phi|_Y = f|_Y$, so $f - \phi \in Y^\perp$. Thus, $\|[f]\|_{X^*/Y^\perp} \leq \|f - (f - \phi)\|_{X^*} = \|\phi\|_{X^*} = \|j([f])\|_{Y^*}$.

Combining these gives $\|f\|_{X^*/Y^\perp} = \|j([f])\|_{Y^*}$, so j is indeed an isometric isomorphism.

Problem 4. *Suppose that X is a separable Hilbert space with inner product $(.,.)_X$, and Y is a linear subspace. Suppose Y is a separable Hilbert space with respect to an inner product $(.,.)_Y$, and the inclusion map $\iota : Y \to X$ is continuous.*

Let $\{x_k\}$ be a sequence in Y such that there exists $C > 0$ with $\|x_k\|_Y \leq C$ for all k, and such that $\{x_k\}$ converges to some $x \in X$ (in the topology of X). Show that $x \in Y$.

You may use that closed balls in a separable Hilbert space are sequentially compact in the topology of weak convergence, i.e. every bounded sequence has a weakly convergent subsequence.

Solution. As $\{x_k\}$ lies in the closed ball of radius C around 0 in Y, which is weakly sequentially compact, $\{x_k\}$ has a Y-weakly convergent subsequence, $\{x_{n(k)}\}$; let $y \in Y$ be the limit. Thus, for all $z \in Y$, $\lim_{k\to\infty}(x_{n(k)}, z)_Y = (y, z)_Y$. As $\iota : Y \to X$ is continuous (with respect to the respective norms), for $w \in X$,

$$(x_{n(k)}, w)_X = (\iota(x_{n(k)}), w)_X = (x_{n(k)}, \iota^* w)_Y,$$

where ι^* is the adjoint of ι, so $\lim_{k\to\infty}(x_{n(k)}, w)_X = (y, \iota^* w)_Y = (\iota y, w)_X$, so $x_{n(k)} \to y$ X-weakly as well. But $\{x_{n(k)}\}$ converges in the X-norm topology to x (i.e. $\|x_{n(k)} - x\|_X \to 0$ as $k \to \infty$), hence X-weakly, i.e. $(x_{n(k)}, w)_X \to (x, w)_X$, so $(x, w)_X = (y, w)_X$ for all $w \in X$, and thus $x = y$ (as $(x - y, x - y)_X = 0$ if we take $w = x - y \in X$). This shows that $x \in Y$.