MATH 205B: PRACTICE MIDTERM

This is a closed book, closed notes, no calculators exam.
There are 4 problems. Solve all of them.

Problem 1. Suppose that $(X, \|.\|_X)$ is a Banach space and Y is a closed linear subspace of X. Suppose that $\|.\|_Y$ is a norm on Y, $(Y, \|.\|_Y)$ is a Banach space, and the inclusion map $\iota : Y \to X$ (with $\iota(y) = y$) is continuous from $(Y, \|.\|_Y)$ to $(X, \|.\|_X)$. Show that the restriction of the norm $\|.\|_X$ to Y is equivalent to $\|.\|_Y$, i.e. there exists a constant $C > 0$ such that $\|y\|_X \leq C\|y\|_Y$ and $\|y\|_Y \leq C\|y\|_X$ for all $y \in Y$.

Problem 2. Let $\ell_{\infty,\mathbb{R}}$ be the Banach space of bounded real valued sequences with the norm $\|\{a_n\}\|_{\ell_{\infty,\mathbb{R}}} = \sup\{|a_n| : \ n \in \mathbb{N}\}$. Show that there is an element λ of $\ell^*_{\infty,\mathbb{R}}$ such that for all sequences $\{a_n\} \in \ell_{\infty,\mathbb{R}}$,

$$\liminf_{n\to\infty} a_n \leq \lambda(\{a_n\}) \leq \limsup_{n\to\infty} a_n.$$

(Hint: first ignore the inequality involving $\liminf_{n\to\infty} a_n$.)

Problem 3. Suppose that $(X, \|.\|)$ is a Banach space and Y is a closed subspace. Let $Y^\perp$ denote the annihilator of Y in X^*, i.e. $Y^\perp = \{f \in X^* : \ f|_Y = 0\}$.

(1) Show that $Y^\perp$ is closed in X^*.
(2) If $f \in X^*$, let $[f]$ denote the image of f in $X^*/Y^\perp$. Let $j : X^*/Y^\perp \to Y^*$ be defined by $j([f]) = f|_Y$. Show that j is well-defined, and is an isometric isomorphism from $X^*/Y^\perp$ onto Y^*.

Problem 4. Suppose that X is a separable Hilbert space with inner product $(.,.)_X$, and Y is a linear subspace. Suppose Y is a separable Hilbert space with respect to an inner product $(.,.)_Y$, and the inclusion map $\iota : Y \to X$ is continuous.

Let $\{x_k\}$ be a sequence in Y such that there exists $C > 0$ with $\|x_k\|_Y \leq C$ for all k, and such that $\{x_k\}$ converges to some $x \in X$ (in the topology of X). Show that $x \in Y$.

You may use that closed balls in a separable Hilbert space are sequentially compact in the weak topology.