MATH 205B: PROBLEM SET 8
DUE THURSDAY, MARCH 10, 2011

Problem 1. (Reed-Simon VI.10) Show that the spectral radius of the Volterra integral operator

$$(Tf)(x) = \int_0^x f(y)\,dy$$

as a map on $C([0,1])$ is equal to 0. What is the norm of T?

Problem 2. (Reed-Simon VI.11) Let $T \in \mathcal{L}(X)$. Prove that $\lim_{n\to\infty} \|T^n\|^{1/n}$ exists and is equal to $\inf_n \|T^n\|^{1/n}$ as follows:

(1) Set $a_n = \log \|T^n\|$, and prove $a_{m+n} \leq a_m + a_n$.
(2) For a fixed positive integer m set $n = mq + r$ where q and r are non-negative integers and $0 \leq r \leq m-1$. Using (1) conclude that
$$\limsup_n \frac{a_n}{n} \leq \frac{a_m}{m}.$$
(3) Prove that $\lim_{n\to\infty} a_n/n = \inf_n a_n/n$ and thus the desired equality.

Problem 3. (Reed-Simon VII.3)

(1) Prove that if A is normal, that is $AA^* = A^*A$, then
$$\|A\| = \sup_{\lambda\in\sigma(A)} |\lambda| \equiv r(A).$$
Hint: Use $\|A\|^2 = \|A^*A\|$ and the formula $r(A) = \lim \|A^n\|^{1/n}$.
(2) Prove that for any polynomial P and any normal operator A, $\|P(A)\| = \sup_{\lambda\in\sigma(A)} |P(\lambda)|$.

Problem 4. Recall that $H^s(\mathbb{S}^1)$ is the Sobolev space of s-times L^2-differentiable functions, and the Fourier series map $\mathcal{F}$ identifies this with $h_s(\mathbb{Z})$ (see Problem 8 on Problem Set 3). Consider $A = -\frac{d^2}{dx^2}$, $A \in \mathcal{L}(H^2(\mathbb{S}^1), L^2(\mathbb{S}^1))$.

(1) Show that if $\lambda \in \mathbb{C}$, $\lambda \neq n^2$, $n \in \mathbb{Z}$, then $\lambda I - A : H^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is a bijection, $R_\lambda(A) = (\lambda I - A)^{-1} \in \mathcal{L}(L^2(\mathbb{S}^1), H^2(\mathbb{S}^1))$, and for $\lambda < 0$, $\|R_\lambda(A)\|_{\mathcal{L}(L^2,L^2)} \leq |\lambda|^{-1}$.
(2) Now suppose $V \in \mathcal{L}(L^2(\mathbb{S}^1))$ (for instance, $V \in C(\mathbb{S}^1)$, considered as a multiplication operator), and let $L = A + V \in \mathcal{L}(H^2(\mathbb{S}^1), L^2(\mathbb{S}^1))$. Fix $\mu > \|V\|_{\mathcal{L}(L^2)}$. Let $K_\lambda = (V - \lambda - \mu) \circ R_{-\mu}(A)$. Show that $K_\lambda \in \mathcal{L}(L^2(\mathbb{S}^1))$ is compact, and $(\mathrm{Id} - K_\lambda)^{-1}$ exists for $\lambda \notin D$, D a discrete set in $\mathbb{C}$.
(3) Show that for $\lambda \notin D$, $\lambda I - L : H^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is a bijection, and $R_\lambda(L) = R_{-\mu}(A)(\mathrm{Id} - K_\lambda)^{-1} \in \mathcal{L}(L^2(\mathbb{S}^1), H^2(\mathbb{S}^1))$, hence the spectrum of L is discrete.

Problem 5. In this problem, we consider operators on $\mathbb{T}^n = (\mathbb{S}^1)^n$. You may use valid results from the Fourier series theory whose analogues for $n = 1$ you have proved on an earlier problem set without giving detailed proofs. Also, $h^s = h^s(\mathbb{Z}^n)$, $s \in \mathbb{R}$, is the space of 'sequences' $\{a_k\} : \mathbb{Z}^n \to \mathbb{C}$ such that

$$\|\{a_k\}\|^2 = \sum_{k\in\mathbb{Z}^n} (1 + |k|^2)^s |a_k|^2 < \infty,$$

and $H^s(\mathbb{T}^n) \subset \mathcal{D}'(\mathbb{T}^n)$ is the corresponding Sobolev space, $H^s(\mathbb{T}^n) = \mathcal{F}^{-1}h^s$. We also write $\langle .,. \rangle$ for the $L^2(\mathbb{T}^n)$ inner product. For a differential operator $P = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha$ we call $p(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ its principal symbol, and say that P is elliptic if $\xi \neq 0$ implies $p \neq 0$.

Consider the operator $P = \sum_{j,k=1}^n D_j a_{jk}(x) D_k + q(x)$ on $\mathbb{T}^n$, $a_{jk}, q \in C^0(\mathbb{T}^n)$ are real valued, and where $(a_{jk}(x))_{j,k=1}^n$ is a positive definite matrix for each $x \in \mathbb{T}^n$. (Equivalently, one can consider P as an operator acting on periodic functions on $\mathbb{R}^n$, hence the notation $D_j = \frac{1}{i}\partial_j$.)

(1) Show that P is elliptic, and for $u, v \in C^\infty(\mathbb{T}^n)$, $\langle Pu, v\rangle = \langle u, Pv\rangle$.
(2) Show that $P : H^1(\mathbb{T}^n) \to H^{-1}(\mathbb{T}^n)$ is continuous, and $\langle Pu, v\rangle = \langle u, Pv\rangle$ holds for $u, v \in H^1(\mathbb{T}^n)$.
(3) Show that there exist $C > 0$ and $C' \geq 0$ such that for $u \in H^1(\mathbb{T}^n)$, $\langle Pu, u\rangle \geq C\|u\|_{H^1}^2 - C'\|u\|_{L^2}^2$, and if $q > 0$, one can take $C' = 0$.
(4) Show that $\|u\|_{H^1} \leq C_1(\|Pu\|_{H^{-1}} + \|u\|_{L^2})$, and if $C' = 0$ then in fact $\|u\|_{H^1} \leq C_1\|Pu\|_{H^{-1}}$.
(5) Show that if $C' = 0$ then $P : H^1 \to H^{-1}$ is invertible.
(6) Show that even if $C' \neq 0$, $\operatorname{Ker} P \subset H^1$ is finite dimensional, $\operatorname{Ran} P \subset H^{-1}$ is closed, and has finite codimension. (Hint: The inclusion map $j : H^1 \to L^2$ is compact!)

Elliptic regularity then shows that if a_{jk} and q are C^∞, and $f \in H^s$, then the solution u of $Pu = f$ satisfies $u \in H^{s+2}$. In particular, for $s > n/2$ (so e.g. $f \in C^\infty(\mathbb{T}^n)$), this implies that $u \in C^2(\mathbb{T}^n)$, and the differential equation holds in the classical sense.

Problem 6. (*This is not to be handed in!*)

(1) Suppose $u \in \mathcal{S}'(\mathbb{R})$ and $x^k u = 0$ for some k (i.e. $u(x^k\phi) = 0$ for all $\phi \in \mathcal{S}(\mathbb{R})$). Show that u is a differentiated delta distribution, namely there exist $a_j \in \mathbb{C}$ such that for all $\phi \in \mathcal{S}(\mathbb{R})$, $u(\phi) = \sum_{j=0}^{k-1} a_j \phi^{(j)}(0)$.
(2) Suppose that $\phi \in \mathcal{S}(\mathbb{R})$ and $\phi^{(j)}(0) = 0$ for $j \leq k$. Show that there exist $\phi_n \in \mathcal{S}(\mathbb{R})$, $n \geq 1$, such that $0 \notin \operatorname{supp}\phi_n$ (i.e. ϕ_n identically 0 near 0), $\phi_n(x) = \phi(x)$ for $|x| \geq 1$ and for all ℓ, $\sum_{j\leq k} \sup_{x\in\mathbb{R}} |x^\ell|\,|\phi_n^{(j)}(x) - \phi^{(j)}(x)| \to 0$.
(3) One says that $x_0 \notin \operatorname{supp} u$ if x_0 has a neighborhood U such that for all $\phi \in \mathcal{S}(\mathbb{R})$ with $\operatorname{supp}\phi \subset U$, $u(\phi) = 0$. Suppose now that $u(\phi) = 0$ if $0 \notin \operatorname{supp}\phi$, so $\operatorname{supp} u \subset \{0\}$. Show that u is a differentiated delta distribution in the sense of (1).

Problem 7. (*This is not to be handed in!*) Suppose that $P = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha \in \operatorname{Diff}_\infty^m(\mathbb{R}^n)$, with $a_\alpha \in C_\infty^\infty(\mathbb{R}^n)$, and let $\langle ., .\rangle$ be the standard L^2-inner product.

(1) Show that there is a unique $P^* \in \operatorname{Diff}_\infty^m(\mathbb{R}^n)$ such that $\langle Pu, v\rangle = \langle u, P^* v\rangle$ for $u, v \in C_c^\infty(\mathbb{R}^n)$.
(2) Suppose that there exist $\Omega \subset \mathbb{R}^n$ open, $r, s \in \mathbb{R}$ and $C > 0$ such that $\|\phi\|_{H^s} \leq C\|P^*\phi\|_{H^r}$ for $\phi \in C_c^\infty(\Omega)$. Show that for any $f \in H^{-s}$, there exists $u \in H^{-r}$ such that $Pu = f$ in Ω, i.e. $Pu(\phi) = f(\phi)$ for $\phi \in C_c^\infty(\Omega)$.
(3) Show that if $T > 0$ then there exists $C > 0$ such that for $\phi \in C_c^\infty(\mathbb{R}^n)$ is supported in $\Omega = \mathbb{R}^{n-1} \times [0, T]$, then $\|\phi\|_{L^2} \leq C\|\partial_n\phi\|_{L^2}$. (This is called a Poincaré inequality.) (Hint: Treat $x_1, \ldots, x_{n-1}$ as parameters, reducing this to a one-dimensional problem.)
(4) Write $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$. Consider the operator $P = D_n^2 - \sum_{j,k=1}^{n-1} D_j a_{jk}(x') D_k$, where a_{jk} is uniformly positive definite, i.e. there exists $c > 0$ such that for all $x' \in \mathbb{R}^{n-1}$, $\sum a_{jk}(x')\xi_j\xi_k \geq c\|\xi\|^2$, $\xi \in \mathbb{R}^{n-1}$. Show that for $T > 0$ there exists $C > 0$ such that for $\phi \in C_c^\infty(\mathbb{R}^{n-1} \times [0, T])$ and for all $j = 1, \ldots, n$, $\|\partial_j\phi\|_{L^2} \leq C\|P\phi\|_{L^2}$. (Hint: consider $\langle (D_n x_n P - P x_n D_n)\phi, \phi\rangle = \langle P\phi, x_n D_n\phi\rangle - \langle x_n D_n\phi, P\phi\rangle$.)
(5) With P as in (4), show that for $T > 0$ and with $\Omega = \mathbb{R}^{n-1} \times [0, T]$, there exists $C > 0$ such that $\phi \in C_c^\infty(\Omega)$, $\|\phi\|_{H^1} \leq C\|P\phi\|_{L^2}$. Use this to prove a solvability result for the wave equation $Pu = f$.

With a bit more effort one could also impose initial conditions for the wave equation; the purpose of this problem to have a concrete example for solvability.