Problem 1. (cf. Reed-Simon Problem V.8 and Theorem V.1) Let $C$ be an absorbing subset of a vector space $V$ with $tx \in C$ if $x \in C$ and $0 \leq t \leq 1$. Let $\rho$ be the Minkowski functional for $C$: $\rho(x) = \inf\{\lambda > 0 : x \in \lambda C\}$. Prove that

1. $\rho(tx) = t\rho(x)$ for $t \geq 0$.
2. $\{x : \rho(x) < 1\} \subset C \subset \{x : \rho(x) \leq 1\}$.
3. $\rho(x + y) \leq \rho(x) + \rho(y)$ if $C$ is convex.
4. $\rho(\lambda x) = |\lambda|\rho(x)$ if $C$ is circled (or balanced).

Using this, prove that if $V$ is a vector space with a Hausdorff topology in which addition and scalar multiplication are separately continuous, and $0$ has a neighborhood base consisting of absorbing, convex and balanced open sets, then $V$ is a locally convex vector space, i.e. the topology arises from a family of seminorms that separate points. (Hint: there is a sketch of the proof in Reed-Simon, ‘proof of Theorem V.1’. You need to write up a detailed proof.)

Problem 2. (Reed-Simon V.9) Prove Theorem V.2.

Problem 3. (cf. Reed-Simon V.17) Let $X, Y$ be locally convex spaces, and let $X^*, Y^*$ be their topological duals. Suppose that $T : X \to Y$ is continuous and linear. Define the adjoint $T^* : Y^* \to X^*$ by $[T^*(y^*)](x) = y^*(Tx)$ for $y^* \in Y^*$, $x \in X$. Prove that

1. If $X^*$ and $Y^*$ are given the $\sigma(X^*, X)$ and $\sigma(Y^*, Y)$ topology (i.e. the weakest topology in which all elements of $X$, resp. $Y$ are continuous), then $T^*$ is continuous.
2. If $X$ and $Y$ are given the $\sigma(X, X^*)$ and $\sigma(Y, Y^*)$ topology, then $T$ is continuous. (This has been proved already if $X$ is a Banach space: strongly continuous maps are weakly continuous.)

Problem 4. Let $D'(S^1)$ denote the topological dual of $C^\infty(S^1)$ with the $\sigma(D'(S^1), C^\infty(S^1))$ topology. One calls elements of $D'(S^1)$ distributions. Let $i : L^1(S^1) \to D'(S^1)$ be the ‘inclusion map’

\[ i(\phi)(\psi) = \int_{S^1} \phi \cdot \psi, \quad \phi \in L^1(S^1), \quad \psi \in C^\infty(S^1). \]

1. Show that $i$ indeed maps into $D'(S^1)$, is injective and continuous. One regards $L^1(S^1)$ as a subset of $D'(S^1)$. (Note: the injectivity of $i|_{C^\infty(S^1)}$ follows easily from Problem 6 on Problem Set 6. Use e.g. Lebesgue’s theorem on Lebesgue points to prove the injectivity on $L^1(S^1)$; this can also be done without measure theory, using convolutions and the density of continuous functions in $L^1$.)

2. Let $\iota = i|_{C^\infty(S^1)}$. Show that $\iota : C^\infty(S^1) \to D'(S^1)$ is continuous.

3. Show that $\frac{d}{dx} : C^\infty(S^1) \to C^\infty(S^1)$ has a continuous extension to a map $\frac{d}{dx} : D'(S^1) \to D'(S^1)$, given by

\[ \frac{d}{dx} u(\phi) = -u(\frac{d\phi}{dx}), \quad u \in D'(S^1), \quad \phi \in C^\infty(S^1). \]

Thus, every distribution, in particular every $L^1$ function, can be differentiated arbitrarily many times, in the sense of distributions.

Problem 5. Let $s(Z)$ denote the set of rapidly decreasing bi-infinite sequences, i.e. sequences $a_n$ such that $\|a_n\|_k = \sup(1 + |n|)^k|a_n| < \infty$ for all $k \geq 0$ integer, and make $s(Z)$ into a Fréchet space (complete locally convex metric space) using these norms.

Let $s'(Z)$ denote the set of polynomially bounded sequences, i.e. sequences $\{a_n\}$ such that for some $k$, $\sup(1 + |n|)^{-k}|a_n| < \infty$. Let $j : s'(Z) \to s(Z)^*$ be given by

\[ j(\{a_n\})(\{b_n\}) = \sum a_n b_n, \quad \{a_n\} \in s'(Z), \quad \{b_n\} \in s(Z). \]

Show that $j$ is a bijection, so $s'(Z)$ can be identified with $s(Z)^*$. We put the $\sigma(s(Z)^*, s(Z))$ topology on $s'(Z)$.