**Problem 1.** (cf. Reed-Simon Problem V.8 and Theorem V.1) Let $C$ be an absorbing subset of a vector space $V$ with $tx \in C$ if $x \in C$ and $0 \leq t \leq 1$. Let $\rho$ be the Minkowski functional for $C$: $\rho(x) = \inf\{\lambda > 0 : x \in \lambda C\}$. Prove that

1. $\rho(tx) = t\rho(x)$ for $t \geq 0$.
2. $\{x : \rho(x) < 1\} \subset C \subset \{x : \rho(x) \leq 1\}$.
3. $\rho(x + y) \leq \rho(x) + \rho(y)$ if $C$ is convex.
4. $\rho(\lambda x) = |\lambda|\rho(x)$ if $C$ is circled (or balanced).

Using this, prove that if $V$ is a vector space with a Hausdorff topology in which addition and scalar multiplication are separately continuous, and 0 has a neighborhood base consisting of absorbing, convex and balanced open sets, then $V$ is a locally convex vector space, i.e. the topology arises from a family of seminorms that separate points. (Hint: there is a sketch of the proof in Reed-Simon, ‘proof of Theorem V.1’. You need to write up a detailed proof.)

**Problem 2.** (Reed-Simon V.9) Prove Theorem V.2.

**Problem 3.** (cf. Reed-Simon V.17) Let $X, Y$ be locally convex spaces, and let $X^*, Y^*$ be their topological duals. Suppose that $T : X \to Y$ is continuous and linear. Define the adjoint $T^* : Y^* \to X^*$ by $[T^*(y^*)](x) = y^*(Tx)$ for $y^* \in Y^*, x \in X$. Prove that

1. If $X^*$ and $Y^*$ are given the $\sigma(X^*, X)$ and $\sigma(Y^*, Y)$ topology (i.e. the weakest topology in which all elements of $X$, resp. $Y$ are continuous), then $T^*$ is continuous.
2. If $X$ and $Y$ are given the $\sigma(X, X^*)$ and $\sigma(Y, Y^*)$ topology, then $T$ is continuous. (This has been proved already if $X$ is a Banach space: strongly continuous maps are weakly continuous.)

**Problem 4.** Let $\mathcal{D}'(S^1)$ denote the topological dual of $C^\infty(S^1)$ with the $\sigma(\mathcal{D}'(S^1), C^\infty(S^1))$ topology. One calls elements of $\mathcal{D}'(S^1)$ distributions. Let $\tilde{i} : L^1(S^1) \to \mathcal{D}'(S^1)$ be the ‘inclusion map’

\[ \tilde{i}(\phi)(\psi) = \int_{S^1} \phi \psi, \ \phi \in L^1(S^1), \ \psi \in C^\infty(S^1). \]

1. Show that $\tilde{i}$ indeed maps into $\mathcal{D}'(S^1)$, is injective and continuous. One regards $L^1(S^1)$ as a subset of $\mathcal{D}'(S^1)$. (Note: the injectivity of $\tilde{i}|_{C^\infty(S^1)}$ follows easily from Problem 6 on Problem Set 6. Use e.g. Lebesgue’s theorem on Lebesgue points to prove the injectivity on $L^1(S^1)$; this can also be done without measure theory, using convolutions and the density of continuous functions in $L^1$.)
2. Let $i = \tilde{i}|_{C^\infty(S^1)}$. Show that $i : C^\infty(S^1) \to \mathcal{D}'(S^1)$ is continuous.
3. Show that $\frac{d}{dx} : C^\infty(S^1) \to C^\infty(S^1)$ has a continuous extension to a map $\frac{d}{dx} : \mathcal{D}'(S^1) \to \mathcal{D}'(S^1)$, given by

\[ \frac{d}{dx} u(\phi) = -u(\frac{d\phi}{dx}), \ u \in \mathcal{D}'(S^1), \ \phi \in C^\infty(S^1). \]

Thus, every distribution, in particular every $L^1$ function, can be differentiated arbitrarily many times, in the sense of distributions.

**Problem 5.** Let $s(\mathbb{Z})$ denote the set of rapidly decreasing bi-infinite sequences, i.e. sequences $a_n$ such that $\|a_n\|_k = \sup_{|n| < k} |a_n| < \infty$ for all $k \geq 0$ integer, and make $s(\mathbb{Z})$ into a Fréchet space (complete locally convex metric space) using these norms.
Let $s'(\mathbb{Z})$ denote the set of polynomially bounded sequences, i.e. sequences $\{a_n\}$ such that for some $k$, $\sup(1 + |n|)^{-k}|a_n| < \infty$. Let $j : s'(\mathbb{Z}) \to s(\mathbb{Z})^*$ be given by

$$j(\{a_n\})(\{b_n\}) = \sum a_n b_n, \quad \{a_n\} \in s'(\mathbb{Z}), \quad \{b_n\} \in s(\mathbb{Z}).$$

Show that $j$ is a bijection, so $s'(\mathbb{Z})$ can be identified with $s(\mathbb{Z})^*$. We put the $\sigma(s(\mathbb{Z})^*, s(\mathbb{Z}))$ topology on $s'(\mathbb{Z})$.

**Problem 6.** Suppose that $X$ is a Banach space, $\Omega \subset \mathbb{R}^n$, $T : \Omega \to \mathcal{L}(X)$ is continuous (in the uniform norm topology). Let $T_z = T(z) \in \mathcal{L}(X)$ for $z \in \Omega$.

1. If $\|T_z\| < 1$ for $z \in \Omega$, show that $(\text{Id} - T_z)^{-1} \in \mathcal{L}(X)$ exists for $z \in \Omega$, and the map $\Omega \ni z \mapsto (\text{Id} - T_z)^{-1} \in \mathcal{L}(X)$ is continuous. (Hint: consider the Neumann series, $\text{Id} + \sum_{j=1}^{\infty} T_z^j$.)

2. If $z_0 \in \Omega$ and $T_{z_0}$ is invertible (but do not assume $\|T_z\| < 1$ on $\Omega$), show that $z_0$ has a neighborhood $O$ in $\Omega$ such that for $z \in O$, $T_z$ is invertible, and the map $O \ni z \mapsto T_z^{-1} \in \mathcal{L}(X)$ is continuous.