MATH 205B: PROBLEM SET 7
DUE FRIDAY, FEBRUARY 28, 2020

Problem 1. (cf. Reed-Simon Problem V.8 and Theorem V.1) Let C be an absorbing subset of a vector space V with $tx \in C$ if $x \in C$ and $0 \le t \le 1$. Let ρ be the Minkowski functional for C: $\rho(x) = \inf\{\lambda > 0 : \ x \in \lambda C\}$. Prove that

(1) $\rho(tx) = t\rho(x)$ for $t \ge 0$.
(2) $\{x : \ \rho(x) < 1\} \subset C \subset \{x : \ \rho(x) \le 1\}$.
(3) $\rho(x+y) \le \rho(x) + \rho(y)$ if C is convex.
(4) $\rho(\lambda x) = |\lambda|\rho(x)$ if C is circled (or balanced).

Using this, prove that if V is a vector space with a Hausdorff topology in which addition and scalar multiplication are separately continuous, and 0 has a neighborhood base consisting of absorbing, convex and balanced open sets, then V is a locally convex vector space, i.e. the topology arises from a family of seminorms that separate points. (Hint: there is a sketch of the proof in Reed-Simon, 'proof of Theorem V.1'. You need to write up a detailed proof.)

Problem 2. (Reed-Simon V.9) Prove Theorem V.2.

Problem 3. (cf. Reed-Simon V.17) Let X, Y be locally convex spaces, and let X^*, Y^* be their topological duals. Suppose that $T : X \to Y$ is continuous and linear. Define the adjoint $T' : Y^* \to X^*$ by $[T'(y^*)](x) = y^*(Tx)$ for $y^* \in Y^*$, $x \in X$. Prove that

(1) If X^* and Y^* are given the $\sigma(X^*, X)$ and $\sigma(Y^*, Y)$ topology (i.e. the weakest topology in which all elements of X, resp. Y are continuous), then T' is continuous.
(2) If X and Y are given the $\sigma(X, X^*)$ and $\sigma(Y, Y^*)$ topology, then T is continuous. (This has been proved already if X is a Banach space: strongly continuous maps are weakly continuous.)

Problem 4. Let $\mathcal{D}'(\mathbb{S}^1)$ denote the topological dual of $C^\infty(\mathbb{S}^1)$ with the $\sigma(\mathcal{D}'(\mathbb{S}^1), C^\infty(\mathbb{S}^1))$ topology. One calls elements of $\mathcal{D}'(\mathbb{S}^1)$ distributions. Let $\tilde{\iota} : L^1(\mathbb{S}^1) \to \mathcal{D}'(\mathbb{S}^1)$ be the 'inclusion map'

$$\tilde{\iota}(\phi)(\psi) = \int_{\mathbb{S}^1} \phi\,\psi,\ \ \phi \in L^1(\mathbb{S}^1),\ \ \psi \in C^\infty(\mathbb{S}^1).$$

(1) Show that $\tilde{\iota}$ indeed maps into $\mathcal{D}'(\mathbb{S}^1)$, is injective and continuous. One regards $L^1(\mathbb{S}^1)$ as a subset of $\mathcal{D}'(\mathbb{S}^1)$. (Note: the injectivity of $\tilde{\iota}|_{C(\mathbb{S}^1)}$ follows easily from Problem 6 on Problem Set 6. Use e.g. Lebesgue's theorem on Lebesgue points to prove the injectivity on $L^1(\mathbb{S}^1)$; this can also be done without measure theory, using convolutions and the density of continuous functions in L^1.)
(2) Let $\iota = \tilde{\iota}|_{C^\infty(\mathbb{S}^1)}$. Show that $\iota : C^\infty(\mathbb{S}^1) \to \mathcal{D}'(\mathbb{S}^1)$ is continuous.
(3) Show that $\frac{d}{dx} : C^\infty(\mathbb{S}^1) \to C^\infty(\mathbb{S}^1)$ has a continuous extension to a map $\frac{d}{dx} : \mathcal{D}'(\mathbb{S}^1) \to \mathcal{D}'(\mathbb{S}^1)$, given by
$$\frac{d}{dx}u(\phi) = -u(\frac{d\phi}{dx}),\ \ u \in \mathcal{D}'(\mathbb{S}^1),\ \ \phi \in C^\infty(\mathbb{S}^1).$$
Thus, every distribution, in particular every L^1 function, can be differentiated arbitrarily many times, in the sense of distributions.

Problem 5. Let $s(\mathbb{Z})$ denote the set of rapidly decreasing bi-infinite sequences, i.e. sequences a_n such that $\|\{a_n\}\|_k = \sup(1+|n|)^k|a_n| < \infty$ for all $k \ge 0$ integer, and make $s(\mathbb{Z})$ into a Fréchet space (complete locally convex metric space) using these norms.

Let $s'(\mathbb{Z})$ denote the set of polynomially bounded sequences, i.e. sequences $\{a_n\}$ such that for some k, $\sup(1+|n|)^{-k}|a_n| < \infty$. Let $j : s'(\mathbb{Z}) \to s(\mathbb{Z})^*$ be given by

$$j(\{a_n\})(\{b_n\}) = \sum a_n b_n,\ \ \{a_n\} \in s'(\mathbb{Z}),\ \ \{b_n\} \in s(\mathbb{Z}).$$

Show that j is a bijection, so $s'(\mathbb{Z})$ can be identified with $s(\mathbb{Z})^*$. We put the $\sigma(s(\mathbb{Z})^*, s(\mathbb{Z}))$ topology on $s'(\mathbb{Z})$.