Problem 1. Suppose that \((X, d)\) is a metric space.

1. Suppose that \(f : [0, \infty) \to [0, \infty)\) satisfies \(f(0) = 0, f(x) > 0\) if \(x > 0\), \(f\) is increasing (i.e. \(x \geq y\) implies \(f(x) \geq f(y)\)) and \(f\) is subadditive: \(f(x + y) \leq f(x) + f(y)\) for all \(x, y\). Show that \(f \circ d : X \times X \to [0, \infty)\) is a metric on \(X\).

2. Suppose that \(f : [0, \infty) \to [0, \infty)\) is \(C^1\) (continuously differentiable), \(f(0) = 0, f'(0) > 0, f'(x) \geq 0\) for all \(x\), and \(f'\) is decreasing (i.e. \(x \leq y\) implies \(f'(x) \geq f'(y)\)). Show that \(f\) is subadditive.

3. Suppose \(d\) and \(d'\) are metrics on \(X\). Show that the topology generated by \(d'\) is weaker than the topology generated by \(d\) (i.e. every open set in \((X, d')\) is open in \((X, d)\)) if and only if given \(\epsilon > 0\) and \(x \in X\) there is \(\delta > 0\) such that

\[
d(x, y) < \delta \Rightarrow d'(x, y) < \epsilon.
\]

Use this to show that with \(f\) as in (2), \(d\) and \(f \circ d\) generate the same topology.

(Such metrics are called equivalent.)

4. Conclude that if \(d\) is a metric on \(X\), then so are \(d' = \frac{d}{1+d}\) and \(d'' = \min(d, 1)\), and these metrics generate the same topology. Note that \(d''(x, y) < 1\) for all \(x, y \in X\), and \(d''(x, y) \leq 1\) for all \(x, y \in X\).

Problem 2. A pseudometric \(\rho\) on a set \(X\) is a map \(\rho : X \times X \to [0, \infty)\) that is symmetric, satisfies the triangle inequality and \(\rho(x, x) = 0\) for all \(x \in X\). (So if \(\rho\) is a pseudometric and \(\rho(x, y) = 0\) implies \(x = y\) then \(\rho\) is a metric.) Let \(\rho_1, \rho_2, \ldots\), be pseudometrics on \(X\) with \(\rho_j \leq 1\). Let

\[
d(x, y) = \sum_{j=1}^{\infty} 2^{-j} \rho_j(x, y).
\]

(1) Show that \(d\) is a pseudometric on \(X\).

(2) Show that if \(x \neq y\) implies that \(\rho_j(x, y) \neq 0\) for some \(j\) then \(d\) is a metric on \(X\), and a sequence \(\{x_n\}\) converges to some \(x \in X\) with respect to \(d\) if and only if given \(\epsilon > 0\) there is \(N\) such that \(n \geq N\) implies \(\rho_j(x_n, x) < \epsilon\). Thus, if the \(\rho_j\) are metrics, then a sequence converges with respect to \(d\) if and only if it converges with respect to \(\rho_j\) for every \(j\).

(3) Show that the topology generated by \(d\) is the weak topology generated by \(\{\rho_n(x, \cdot) : x \in X, n \in \mathbb{Z}^+\}\), i.e. the weakest topology in which these functions are all continuous.

(4) Now suppose that \(X\) is a vector space and each \(d_j\) is a translation invariant metric, i.e. \(d_j(x + z, y + z) = d_j(x, y)\) for all \(x, y, z \in X\). Let \(\rho_j\) be translation invariant metrics equivalent to \(d_j\) with \(\rho_j \leq 1\). Show that a sequence \(\{x_n\}\) is Cauchy with respect to \(d\) if and only if it is Cauchy with respect to every \(d_j\).

(5) Now suppose that \(X_1, X_2, \ldots\) are vector spaces, \(X_1 \supset X_2 \supset \ldots\) and \(X = \cap_{k=1}^{\infty} X_k\). Let \(d_k\) be translation invariant metrics on \(X_k\), and suppose that the inclusion maps \(\iota_k : X_k \to X_{k-1}\) are all continuous. Show that if \((X_k, d_k)\) is complete for every \(k\) then \((X, d)\) is complete.
(6) Let $C^\infty(S^1)$ denote the set of complex valued infinitely differentiable functions on $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$. Let $d_k$ be the metric given by the $C^k$ norm:

$$
\|f\|_{C^k} = \sum_{m=0}^{k} \sup_{x \in S^1} |f^{(m)}(x)|.
$$

Let $d$ be the corresponding metric on $C^\infty(S^1)$. Show that $C^\infty(S^1)$ is a complete metric space in which sequences $\{x_n\}$ converge, resp. are Cauchy, if and only if they converge, resp. are Cauchy, in every $C^k$. (Thus, convergence of a sequence $\{f_n\}$ is just the uniform convergence of all derivatives $\{f_n^{(k)}\}$.)

**Problem 3.** Suppose that $(X, \mathcal{T})$ is a compact topological space and let $F = \{f_1, f_2, \ldots\}$ be a countable collection of continuous real valued functions on $X$ that separate points (i.e. for $x \neq y$ there is $j$ such that $f_j(x) \neq f_j(y)$).

1. Show that without loss of generality we may assume $|f_j(x)| < 1/2$ for all $x$; assume this from now on.
2. Let $\rho_j(x, y) = |f_j(x) - f_j(y)|$. Show that $\rho_j$ is a pseudometric. With $d$ given by Equation (1) of Problem 2, show that $d$ is a metric on $X$; let $\mathcal{T}_d$ denote the metric topology on $X$.
3. Show that $\mathcal{T}_d = \mathcal{T}$, i.e. $(X, \mathcal{T})$ is metrizable. (Hint: show that $\mathcal{T}_d$ is Hausdorff and it is the $F$-weak topology, hence is weaker than $\mathcal{T}$.)
4. Suppose $Y$ is a separable Banach space. Show that the closed unit ball $B$ in $Y^*$ is metrizable in the weak-* topology.

**Problem 4.** (Reed-Simon IV.19) Suppose that $f : [-1/2, 1/2] \to \mathbb{R}$ is a continuous function with $f(1/2) = f(-1/2) = 0$. Let $s_k : [-1, 1] \to \mathbb{R}$ be continuous, $s_k \geq 0$, $\int_{-1}^{1} s_k (x) \, dx = 1$, and for each $\delta > 0$,

$$
\lim_{k \to \infty} \int_{1 \geq |x| \geq \delta} s_k(x) = 0.
$$

Let $f_k : [-1/2, 1/2] \to \mathbb{R}$ be given by

$$
f_k(x) = \int_{-1/2}^{1/2} s_k(x - y) f(y) \, dy.
$$

Show that $f_k \to f$ uniformly on $[-1/2, 1/2]$.

**Problem 5.** (Reed-Simon IV.20) Let

$$
s_k = (I_k)^{-1}(1 - x^2)^k, \text{ where } I_k = \int_{-1}^{1} (1 - x^2)^k \, dx.
$$

Using the previous problem, prove that any continuous function on $[-1/4, 1/4]$ is the limit of polynomials uniformly on $[-1/4, 1/4]$. (This is the classical Weierstrass approximation theorem.)

**Problem 6.** (cf. Reed-Simon V.61) For $\phi \in C(X)$, $X$ a topological space, supp $\phi$ is the closure of $\{x : \phi(x) \neq 0\}$.

Show that if $x \in \mathbb{R}^n$ and $\delta > 0$ then there exists $\phi \in C^\infty(\mathbb{R}^n)$ such that $\phi \geq 0$, $\phi(x) > 0$ and supp $\phi \subset \{y : \|y - x\| < \delta\} = B(x, \delta)$. (Hint: Show first that the function $\chi : \mathbb{R} \to \mathbb{R}$ defined by $\chi(t) = 0$ for $t \leq 0$, $\chi(t) = e^{-1/t}$, $t > 0$, is $C^\infty$.)