MATH 205B: PROBLEM SET 4
DUE FRIDAY, FEBRUARY 7, 2020

Problem 1. Suppose that \((X, d)\) is a metric space. A \(G_\delta\) set in \(X\) is a subset \(A\) of \(X\) such that there exist open sets \(O_n, n = 1, 2, \ldots\), such that \(\cap_{n=1}^\infty O_n = A\).

1) Suppose \(f : X \to \mathbb{R}\). Show that the set of points \(x\) at which \(f\) is continuous is a \(G_\delta\) set.

2) Suppose that \(X\) is complete. Suppose also that \(A \subset X\), and \(A\) and \(A^c = X \setminus A\) are both dense in \(X\). Show that at most one of \(A\) and \(A^c\) is a \(G_\delta\) set.

3) Show that if \(X = [0, 1]\), there is no function \(f : X \to \mathbb{R}\) that is continuous at all rationals and discontinuous at all irrationals.

Problem 2. (Reed-Simon IV.30)

1) Show that every metric space is normal.

2) Prove that every closed set in a metric space is a \(G_\delta\).

Problem 3. Suppose \(X\) is an infinite dimensional Banach space. Show that the closed unit ball, \(B = \{ x \in X : \|x\| \leq 1 \}\), is not compact (in the norm topology).

Problem 4. Suppose that \((X, \|\cdot\|_X)\) is a normed vector space, \(M\) and \(N\) are (not necessarily closed) subspaces equipped with norms \(\|\cdot\|_M\), resp. \(\|\cdot\|_N\) such that the identity maps \((M, \|\cdot\|_M) \to (M, \|\cdot\|_N)\), resp. \((N, \|\cdot\|_N) \to (N, \|\cdot\|_X)\) are continuous. Let \(M + N\) be the algebraic sum: \(M + N = \{ m + n : m \in M, n \in N \}\). For \(x \in M + N\), let \(\|x\|_{M+N} = \inf \{ \|m\|_M + \|n\|_N : m \in M, n \in N, x = m + n \}\).

1) Show that \(\|\cdot\|_{M+N}\) is a norm on \(M + N\).

2) Show that if \((M, \|\cdot\|_M)\) and \((N, \|\cdot\|_N)\) are complete then \((M + N, \|\cdot\|_{M+N})\) is complete.

Problem 5. (Reed-Simon III.15) Let \(\mathcal{H}\) be a separable Hilbert space with an orthonormal basis \(\{ x_n \}_{n=1}^\infty\). Let \(\{ y_n \}\) be a sequence in \(\mathcal{H}\) and prove that the following two statements are equivalent.

1) \(\lim_{n \to \infty} (x, y_n) = 0\) for all \(x \in \mathcal{H}\).

2) \(\lim_{n \to \infty} (x_m, y_n) = 0\) for each \(m = 1, 2, \ldots\), and \(\{ \|y_n\| \}_{n=1}^\infty\) is bounded.

If either one of these conditions holds, one says that \(\{ y_n \}\) converges to \(0\) weakly, and more generally, we say that \(\{ y_n \}\) converges to \(y \in \mathcal{H}\) weakly if \(\{ y_n - y \}\) converges to \(0\) weakly. (In a week we will actually talk about ‘weak topologies’, so far this is just a definition without a topology (i.e. a notion of open sets) behind it.)

Problem 6. Let \(\mathcal{H}\) be a separable infinite dimensional Hilbert space, and let \(S\) denote the unit sphere \(S = \{ x \in \mathcal{H} : \|x\| = 1\}\). If \(\{ y_n \}\) be a sequence in \(\mathcal{H}\), we say that \(y_n \to y\) weakly if \(\lim_{n \to \infty} (x, y_n) = (x, y)\) for all \(x \in \mathcal{H}\).

1) Suppose \(y_n \in S\) for all \(n\), and \(y_n \to y\) \(\to \in \mathcal{H}\) weakly. Show that \(\|y\| \leq 1\).

2) Suppose that \(y \in \mathcal{H}\) and \(\|y\| \leq 1\). Show that there exists a sequence \(\{ y_n \}_{n=1}^\infty\) with \(y_n \in S\) for all \(n\) such that \(y_n \to y\) weakly.

Problem 7. Suppose that \(\mathcal{H}\) is a separable Hilbert space, \(\{ y_n \}_{n=1}^\infty\) is a sequence in \(\mathcal{H}\) and \(\|y_n\| \leq 1\) for all \(n\). Show that there is a subsequence \(\{ y_{n(k)} \}_{k=1}^\infty\) and a \(y \in \mathcal{H}\) with \(\|y\| \leq 1\) such that for all \(x \in \mathcal{H}\),

\[
\lim_{k \to \infty} (x, y_{n(k)}) = (x, y).
\]

In other words, show that the closed unit ball in a separable Hilbert space is sequentially compact in the ‘topology of weak convergence’. (This will be a consequence of the Banach-Alaoglu theorem in the general Banach space setting, using that \(\mathcal{H}^* = \mathcal{H}\).)