MATH 205B: PROBLEM SET 4
DUE FRIDAY, FEBRUARY 7, 2020

Problem 1. Suppose that (X, d) is a metric space. A G_δ set in X is a subset A of X such that there exist open sets O_n, $n = 1, 2, \ldots$, such that $\cap_{n=1}^{\infty} O_n = A$.

(1) Suppose $f : X \to \mathbb{R}$. Show that the set of points x at which f is continuous is a G_δ set.
(2) Suppose that X is complete. Suppose also that $A \subset X$, and A and $A^c = X \setminus A$ are both dense in X. Show that at most one of A and A^c is a G_δ set.
(3) Show that if $X = [0, 1]$, there is no function $f : X \to \mathbb{R}$ that is continuous at all rationals and discontinuous at all irrationals.

Problem 2. (Reed-Simon IV.30)

(1) Show that every metric space is normal.
(2) Prove that every closed set in a metric space is a G_δ.

Problem 3. Suppose X is an infinite dimensional Banach space. Show that the closed unit ball, $B = \{x \in X : \ \|x\| \leq 1\}$, is not compact (in the norm topology).

Problem 4. Suppose that $(X, \|.\|_X)$ is a normed vector space, M and N are (not necessarily closed) subspaces equipped with norms $\|.\|_M$, resp. $\|.\|_N$ such that the identity maps $(M, \|.\|_M) \to (M, \|.\|_X)$, resp. $(N, \|.\|_N) \to (N, \|.\|_X)$ are continuous. Let $M + N$ be the algebraic sum: $M + N = \{m + n \in X : \ m \in M, \ n \in N\}$. For $x \in M + N$, let

$$\|x\|_{M+N} = \inf\{\|m\|_M + \|n\|_N : \ m \in M, \ n \in N, \ x = m + n\}.$$

(1) Show that $\|.\|_{M+N}$ is a norm on $M + N$.
(2) Show that if $(M, \|.\|_M)$ and $(N, \|.\|_N)$ are complete then $(M + N, \|.\|_{M+N})$ is complete.

Problem 5. (Reed-Simon III.15) Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{x_n\}_{n=1}^{\infty}$. Let $\{y_n\}$ be a sequence in $\mathcal{H}$ and prove that the following two statements are equivalent.

(1) $\lim_{n\to\infty}(x, y_n) = 0$ for all $x \in \mathcal{H}$.
(2) $\lim_{n\to\infty}(x_m, y_n) = 0$ for each $m = 1, 2, \ldots$, and $\{\|y_n\|\}_{n=1}^{\infty}$ is bounded.

If either one of these conditions holds, one says that $\{y_n\}$ converges to 0 weakly, and more generally, we say that $\{y_n\}$ converges to $y \in \mathcal{H}$ weakly if $\{y_n - y\}$ converges to 0 weakly. (In a week we will actually talk about 'weak topologies', so far this is just a definition without a topology (i.e. a notion of open sets) behind it.)

Problem 6. Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space, and let S denote the unit sphere $S = \{x \in \mathcal{H} : \ \|x\| = 1\}$. If $\{y_n\}$ be a sequence in $\mathcal{H}$, we say that $y_n \to y$ *weakly* if $\lim_{n\to\infty}(x, y_n) = (x, y)$ for all $x \in \mathcal{H}$.

(1) Suppose $y_n \in S$ for all n, and $y_n \to y \in \mathcal{H}$ weakly. Show that $\|y\| \leq 1$.
(2) Suppose that $y \in \mathcal{H}$ and $\|y\| \leq 1$. Show that there exists a sequence $\{y_n\}_{n=1}^{\infty}$ with $y_n \in S$ for all n such that $y_n \to y$ weakly.

Problem 7. Suppose that $\mathcal{H}$ is a separable Hilbert space, $\{y_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{H}$ and $\|y_n\| \leq 1$ for all n. Show that there is a subsequence $\{y_{n(k)}\}_{k=1}^{\infty}$ and a $y \in \mathcal{H}$ with $\|y\| \leq 1$ such that for all $x \in \mathcal{H}$,

$$\lim_{k\to\infty}(x, y_{n(k)}) = (x, y).$$

In other words, show that the closed unit ball in a separable Hilbert space is sequentially compact in the 'topology of weak convergence'. (This will be a consequence of the Banach-Alaoglu theorem in the general Banach space setting, using that $\mathcal{H}^* = \mathcal{H}$.)