MATH 205B: PROBLEM SET 3
DUE MONDAY, FEBRUARY 3, 2020

Problem 1. (Reed-Simon III.2)

(1) Prove that ℓ_p $(1 \leq p < \infty)$ and c_0 are separable, but ℓ_∞ is not.
(2) Prove that $s \subset \ell_p$ for all p.

Problem 2. (Reed-Simon III.3) Prove that a normed linear space is complete if and only if every absolutely summable sequence is summable. (Hint for the "if" part: to show that a Cauchy sequence converges, it suffices to show that a subsequence converges.)

Problem 3. (Reed-Simon III.7) Prove that $\ell_\infty = \ell_1^*$, but $\ell_\infty^* \neq \ell_1$ by using the Hahn-Banach theorem. (More precisely, for the last part, note that ℓ_1 can be identified with a subset of ℓ_∞^* by $\{a_n\}(\{b_n\}) = \sum_{n=1}^\infty a_n b_n$ for $\{a_n\} \in \ell_1$, $\{b_n\} \in \ell_\infty$. Show that *not* every element of ℓ_∞^* is of this form.)

Problem 4. (Reed-Simon III.10) Prove that if Z is a subspace of a normed linear space X, and $y \in X$ has distance d from Z, then there exists $\Lambda \in X^*$ such that $\|\Lambda\| \leq 1$, $\Lambda(y) = d$ and $\Lambda(z) = 0$ for all $z \in Z$. (Note: the distance referred to above is $d = \inf\{\|z - y\| : \ z \in Z\}$.)

Problem 5. Suppose that X is a normed vector space over $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$, and $\lambda : X \to \mathbb{F}$ is linear. Show that λ is continuous if and only if $N = \lambda^{-1}(\{0\}) = \{x \in X : \ \lambda(x) = 0\}$ is closed.

Problem 6. (Reed-Simon III.12) Let $X = C([0,1])$, $M = \{f : \ f(0) = 0\}$. Show that the Banach quotient space is $X/M = \mathbb{C}$.

Problem 7. (Reed-Simon III.23) Let X be a Hilbert space and $\mathcal{M}$ a closed subspace. Show that the restriction of the natural map $\pi : X \to X/\mathcal{M}$ is an isomorphism of $\mathcal{M}^\perp$ and $X/\mathcal{M}$.

Problem 8. Suppose that X is a normed vector space, M a closed subspace and $\pi : X \to X/M$ the projection. Show that topology induced by the standard norm on the quotient space,

$$\|[x]\|_{X/M} = \inf\{\|x + m\|_X : \ m \in M\},$$

is the usual quotient topology, i.e. that $O \subset X/M$ is open in X/M if and only if $\pi^{-1}(O)$ is open in X.

Problem 9. For $s \in \mathbb{R}$, let h_s be the set of bi-infinite sequences $\{a_n\}$ with $\{(1 + |n|)^s a_n\} \in \ell_2(\mathbb{Z})$.

(1) Show that the following inner products on h_s:

$$(\{a_n\}, \{b_n\})_{s,1} = (\{(1 + |n|)^s a_n\}, \{(1 + |n|)^s b_n\})_{\ell_2(\mathbb{Z})}$$

and

$$(\{a_n\}, \{b_n\})_{s,2} = (\{(1 + n^2)^{s/2} a_n\}, \{(1 + n^2)^{s/2} b_n\})_{\ell_2(\mathbb{Z})}$$

are well defined and give equivalent norms. (So as far as topological questions, such as convergence, are concerned, you can use either; the second is the standard inner product.)

(2) Show that h_s is complete.

(3) Suppose that $s > 0$. Show that if $C > 0$ and $\{y_m\}_{m=1}^{\infty}$ is a sequence in h_s with $\|y_m\|_{h_s} \leq C$ for all m, then it has a subsequence $\{y_{m(k)}\}_{k=1}^{\infty}$ such that $\{y_{m(k)}\}$ converges to some $y \in \ell_2(\mathbb{Z})$ in $\ell_2(\mathbb{Z})$. (This states that the $\ell_2(\mathbb{Z})$-closure of a ball in h_s is a compact subset of $\ell_2(\mathbb{Z})$ in the standard $\ell_2(\mathbb{Z})$ topology, and one says that the inclusion of h_s, $s > 0$, into $\ell_2(\mathbb{Z})$ is compact.)

(4) For $s \geq 0$, let $H^s(\mathbb{S}^1)$ be the subspace of $L^2(\mathbb{S}^1)$ consisting of $f \in L^2(\mathbb{S}^1)$ with $\mathcal{F}f \in h^s$, and with inner product

$$(f, g)_{H^s} = (\mathcal{F}f, \mathcal{F}g)_{s,2}.$$

(Thus, the Fourier series map $\mathcal{F} : L^2(\mathbb{S}^1) \to \ell_2(\mathbb{Z})$ restricts to a Hilbert space isomorphism $\mathcal{F} : H^s(\mathbb{S}^1) \to h_s$.) Show that for $s > 0$, the L^2-closure of the unit ball in H^s is compact in L^2.

(5) Problem 4 (1) on Problem Set 2 implies that $C^1(\mathbb{S}^1) \subset H^1(\mathbb{S}^1)$. Show that $C^1(\mathbb{S}^1)$ is a dense subset of $H^1(\mathbb{S}^1)$, and in fact even finite linear combinations of the e_n are dense in $H^1(\mathbb{S}^1)$.

(6) If $f \in C^1(\mathbb{S}^1) = C^1_p([0, 2\pi])$, you have shown that $(e_n, f') = in(e_n, f)$. Using this, show that for $f \in C^1(\mathbb{S}^1)$, $\|f'\|_{L^2} \leq C\|f\|_{H^1}$, and hence differentiation extends to a bounded linear map $\frac{d}{dx} : H^1(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$, and the extension is unique.

(7) Show that for $f \in H^1(\mathbb{S}^1)$, $\|f\|^2_{H^1(\mathbb{S}^1)} = \|f\|^2_{L^2} + \|\frac{d}{dx}f\|^2_{L^2}$. (This is why $(.,.)_{s,2}$ is the standard inner product on h_s.)

(8) Show that for $a \in C^1(\mathbb{S}^1)$, the multiplication operator by a, $M_a : C^1(\mathbb{S}^1) \to C^1(\mathbb{S}^1)$, $M_a f(x) = a(x)f(x)$, extends to a unique continuous linear map $H^1(\mathbb{S}^1) \to H^1(\mathbb{S}^1)$. (It can be shown that under these assumptions $M_a : H^s(\mathbb{S}^1) \to H^s(\mathbb{S}^1)$ for $0 \leq s \leq 1$.)