MATH 205B: PROBLEM SET 2
DUE FRIDAY, JANUARY 24, 2020

Problem 1. (cf. Reed-Simon, II.4) Suppose V is an inner product space either over $\mathbb{R}$ or over $\mathbb{C}$.

(1) Prove that the inner product can be recovered from the norm by the polarization identity:
$$(x, y) = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2)$$
if the field is $\mathbb{R}$, and
$$(x, y) = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2) - i\frac{1}{4}(\|x + iy\|^2 - \|x - iy\|^2)$$
if the field is $\mathbb{C}$.

(2) Prove that a normed vector space $(V, \|.\|)$ is an inner product space (with the induced norm being $\|.\|$) if and only if the norm satisfies the parallelogram law:
$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$
for all $x, y \in V$. (Hint: define an 'inner product' by the polarization identity, check its properties, in particular additivity: $(x, y + z) = (x, y) + (x, z)$. Homogeneity, i.e. $(x, cy) = c(x, y)$, follows for c rational.)

(3) Conclude that the standard norms on ℓ_p and $L^p(\mathbb{R})$, $p \neq 2$, do not arise from inner products.

Problem 2. (Reed-Simon II.8) Complete the proof of the Corollary to the Riesz Lemma (Theorem II.4), i.e. that if $B : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is a sesquilinear form, linear in the second slot, conjugate linear in the first slot, and there is $C \geq 0$ such that $|B(x, y)| \leq C\|x\| \|y\|$ for all $x, y \in \mathcal{H}$, then there is a unique $A \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ such that $B(x, y) = (Ax, y)$ for all $x, y \in \mathcal{H}$. The norm of A is the smallest C for which $|B(x, y)| \leq C\|x\| \|y\|$ holds for all $x, y \in \mathcal{H}$.

Problem 3. Let $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$ denote the circle, which can be thought of as $[0, 2\pi]$ with 0 and 2π identified. Let $C^k(\mathbb{S}^1)$ (or $C^k_p(\mathbb{R})$, or $C^k_p([0, 2\pi])$) denote the space of 2π-periodic C^k functions on $\mathbb{R}$. Let $M : C^k(\mathbb{S}^1) \to C^k(\mathbb{S}^1)$ denote the multiplication operator by e^{ix}: $(Mf)(x) = e^{ix}f(x)$, and for $k \geq 1$ (including $k = \infty$) let $\frac{d}{dx} : C^k(\mathbb{S}^1) \to C^{k-1}(\mathbb{S}^1)$ be the usual derivative.

Suppose that $T : C^\infty(\mathbb{S}^1) \to C^\infty(\mathbb{S}^1)$ is a linear map (no continuity of any kind is assumed!) and T satisfies $T\frac{d}{dx} = \frac{d}{dx}T$ and $TM = MT$ (on $C^\infty(\mathbb{S}^1)$). Show that there exists $c \in \mathbb{C}$ such that $T = c\,\mathrm{Id}$, i.e. $Tf = cf$ for all $f \in C^\infty(\mathbb{S}^1)$.

(Hint: First show that if $y \in \mathbb{S}^1$ and $f(y) = 0$ then $(Tf)(y) = 0$; to do so, use Taylor's theorem. This shows that T is multiplication by a function; show that this function is C^∞ and its derivative vanishes.)

Problem 4. Let $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$ denote the circle, which can be thought of as $[0, 2\pi]$ with 0 and 2π identified. Let $C^k(\mathbb{S}^1)$ (or $C^k_p(\mathbb{R})$, or $C^k_p([0, 2\pi])$) denote the space of 2π-periodic C^k functions on $\mathbb{R}$; let $L^p(\mathbb{S}^1)$ be the space of 2π-periodic elements of $L^p_{\mathrm{loc}}(\mathbb{R})$. Let $e_n \in C^\infty_p([0,2\pi]) = C^\infty(\mathbb{S}^1)$ be given by $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$, $n \in \mathbb{Z}$. Let $\ell_2(\mathbb{Z})$ denote the space of square summable bi-infinite sequences, i.e. elements of $\ell_2(\mathbb{Z})$ are maps $a : \mathbb{Z} \to \mathbb{C}$, usually denoted by $\{a_n\} = \{a_n\}_{n=-\infty}^{\infty}$, with $\sum |a_n|^2 < \infty$, and the inner product

$$(a_n, b_n) = \sum_{n=-\infty}^{\infty} \overline{a_n}\, b_n.$$

Let $\mathcal{F}$ denote the Fourier series map, so

$$\mathcal{F} : L^2(\mathbb{S}^1) \to \ell_2(\mathbb{Z}),\ (\mathcal{F}f)_n = (e_n, f)_{L^2},$$

and let $\mathcal{F}^{-1}$ denote the map

$$\mathcal{F}^{-1} : \ell_2(\mathbb{Z}) \to L^2(\mathbb{S}^1),\ \mathcal{F}^{-1}\{c_n\} = \sum_{n\in\mathbb{Z}} c_n e_n.$$

(1) If $f \in C^1(\mathbb{S}^1) = C^1_p([0,2\pi])$, show that $(e_n, f') = in(e_n, f)$, and hence $\mathcal{F}\frac{d}{dx} = in\mathcal{F}$ on $C^1(\mathbb{S}^1)$.
(2) Show that $\mathcal{F} : C^1(\mathbb{S}^1) \to \ell_1(\mathbb{Z})$, and is continuous.
(3) Show that $\mathcal{F}^{-1} : \ell_1(\mathbb{Z}) \to C^0(\mathbb{S}^1)$, with the series defining $\mathcal{F}^{-1}$ coverging uniformly, and $\mathcal{F}^{-1}$ is continuous as such a map. Conclude that for $f \in C^1(\mathbb{S}^1)$ the Fourier series of f converges uniformly.
(4) Show that if $\{nc_n\} \in \ell_1(\mathbb{Z})$, then $\mathcal{F}^{-1}\{c_n\} \in C^1(\mathbb{S}^1)$ and $\frac{d}{dx}\mathcal{F}^{-1}\{c_n\} = \mathcal{F}^{-1}\{inc_n\}$.
(5) Show that for $f \in L^2(\mathbb{S}^1)$, $\mathcal{F}(e^{ix}f)_n = (\mathcal{F}f)_{n-1}$, and show that for $\{c_n\} \in \ell_2(\mathbb{Z})$, $\mathcal{F}^{-1}\{c_{n-1}\} = e^{ix}\mathcal{F}^{-1}\{c_n\}$.
(6) Let $s(\mathbb{Z})$ denote the space of rapidly decreasing bi-infinite sequences, i.e. maps $a : \mathbb{Z} \to \mathbb{C}$ such that for all $k \in \mathbb{N}$, $(1+|n|)^k|a_n|$ is bounded. Show that $\mathcal{F} : C^\infty(\mathbb{S}^1) \to s(\mathbb{Z})$ and $\mathcal{F}^{-1} : s(\mathbb{Z}) \to C^\infty(\mathbb{S}^1)$.
(7) Show that there exists $c \in \mathbb{C}$ such that for $f \in C^\infty(\mathbb{S}^1)$, $\mathcal{F}^{-1}\mathcal{F}f = cf$, and show that in fact $c = 1$.
(8) Show that $\mathcal{F}^{-1}\mathcal{F} = \mathrm{Id}$ on $L^2(\mathbb{S}^1)$.