MATH 205B: PROBLEM SET 1
DUE FRIDAY, JANUARY 17, 2020

Problem 1. (Reed-Simon, Sec.I, no. 3) Suppose that x_n is a Cauchy sequence in a metric space (X, ρ). Suppose that for some subsequence, $x_{n(i)}$, $\lim_{i\to\infty} x_{n(i)} = x_\infty$. Prove that $\lim_{n\to\infty} x_n = x_\infty$.

Problem 2. (Reed-Simon, Sec.I, no. 4) Let x_n be a sequence in a metric space and let x_∞ be given. Suppose that every subsequence of x_n has a sub-subsequence converging to x_∞. Prove that $\lim_{n\to\infty} x_n = x_\infty$.

Problem 3. (cf. Reed-Simon, Sec.I, no. 5) Suppose (M, d) is an incomplete metric space. Show that there is a complete metric space $(\tilde{M}, \tilde{d})$ and an isometry $\iota : M \to \tilde{M}$ such that $\iota(M)$ is dense in $\tilde{M}$.

Show also that $(\tilde{M}, \tilde{d})$ is essentially unique, i.e. if $(\tilde{M}', \tilde{d}')$ is another space with these properties then there is an invertible map $j : \tilde{M} \to \tilde{M}'$ which is an isometry and such that $\iota' = j \circ \iota$.

(N.B. It is easy to show that if (M, d) is a normed vector space $(V, \|.\|)$, then the completion is also a normed vector space, with the new norm being an extension of the old one. Cf. Problem 6 below. You do not need to write this up though.)

Problem 4. (cf. Reed-Simon, Sec.I, no. 7) Suppose that T is a linear transformation between two normed vector spaces. Show that the following are equivalent:

(1) T is continuous at one point,
(2) T is continuous at all points,
(3) T is uniformly continuous,
(4) T is bounded.

Problem 5. (Reed-Simon, Sec.I, no. 32) Let $F \in C([0,1] \times [0,1])$ and consider the map $\mathcal{F} : C([0,1]) \to C([0,1])$ given by

$$(\mathcal{F}f)(x) = \int_0^1 F(x,y)\, f(y)\, dy.$$

Prove that $\{\mathcal{F}f :\ \|f\|_\infty \le 1\}$ is an equicontinuous family so that any given sequence f_n with $\|f_n\| \le 1$ for all n has a subsequence $f_{n(i)}$ with $\mathcal{F}f_{n(i)}$ uniformly convergent.

Problem 6. (Reed-Simon, Sec.II, no. 1)

(1) Let V be an inner product space. Prove that the inner product can be extended to the completion, $\tilde{V}$, as follows. First, show that if $x, y \in \tilde{V}$, $x_n, y_n \in V$, $\lim x_n = x$, $\lim y_n = y$ then (x_n, y_n) converges. Define $(x, y) = \lim_{n\to\infty}(x_n, y_n)$, and show that it is independent of which convergent sequences are chosen. Finally show that $(.,.)$ has the right properties.
(2) Prove the statement in (1) by applying the B.L.T. theorem twice.

Problem 7. (Reed-Simon, Sec.II, no. 6) Let $\mathcal{M}$ be any linear subspace of a Hilbert space $\mathcal{H}$. Prove that $\mathcal{M}^\perp$ is a closed linear subspace and $\overline{\mathcal{M}} = (\mathcal{M}^\perp)^\perp$, with the bar denoting closure.

Problem 8. (Reed-Simon, Sec.III, no. 4) Show that all norms on $\mathbb{R}^n$ are equivalent. (Hint: Use the fact that the unit sphere is compact in the Euclidean topology.)

Problem 9. (Reed-Simon, Sec.II, no. 9) Let $\mathcal{M}$ be a subspace of a Hilbert space $\mathcal{H}$. Let $f : \mathcal{M} \to \mathbb{C}$ be a linear functional on $\mathcal{M}$ with bound C. Prove that there is a unique extension of f to a linear functional on $\mathcal{H}$ with the same bound. (N.B. The existence part would follow from Hahn-Banach for Hilbert spaces, but is easy to do directly: this is an example of the claim that Hilbert spaces are simpler to deal with than Banach spaces.)

Problem 10. (Least squares approximation, Reed-Simon II.5) Let V be an inner product space, and $\{x_n\}_{n=1}^N$ be an orthonormal set. Prove that

$$\|x - \sum_{n=1}^{N} c_n x_n\|$$

is minimized by choosing $c_n = (x_n, x)$.

Finally if you want a head start on Problem Set 2 (and you have time):

Problem 11. (cf. Reed-Simon, II.4, *Hand in with Problem Set 2!*) Suppose V is an inner product space either over $\mathbb{R}$ or over $\mathbb{C}$.

(1) Prove that the inner product can be recovered from the norm by the polarization identity:

$$(x, y) = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2)$$

if the field is $\mathbb{R}$, and

$$(x, y) = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2) - i\frac{1}{4}(\|x + iy\|^2 - \|x - iy\|^2)$$

if the field is $\mathbb{C}$.

(2) Prove that a normed vector space $(V, \|.\|)$ is an inner product space (with the induced norm being $\|.\|$) if and only if the norm satisfies the parallelogram law:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for all $x, y \in V$. (Hint: define an 'inner product' by the polarization identity, check its properties, in particular additivity: $(x, y + z) = (x, y) + (x, z)$. Homogeneity, i.e. $(x, cy) = c(x, y)$, follows for c rational.)

(3) Conclude that the standard norms on ℓ_p and $L^p(\mathbb{R})$, $p \neq 2$, do not arise from inner products.