Problem 1. (10 points) Do Problem 8.5.

Problem 2. (10 points) Do Problem 8.11.

Problem 3. (10 points) Do Problem 8.12.


Problem 5. (30 points) Recall that $H^s(S^1)$ is the Sobolev space of $s$-times $L^2$-differentiable functions, and the Fourier series map $F$ identifies this with $h_s(Z)$ (see Problem 5 on Problem Set 7). Consider $A = -\frac{d^2}{dx^2}$, $A \in \mathcal{L}(H^2(S^1), L^2(S^1))$.

1. Show that if $\lambda \in \mathbb{C}$, $\lambda \neq n^2$, $n \in \mathbb{Z}$, then $\lambda I - A : H^2(S^1) \rightarrow L^2(S^1)$ is a bijection, $R_\lambda(A) = (\lambda I - A)^{-1} \in \mathcal{L}(L^2(S^1), H^2(S^1))$, and for $\lambda < 0$, $\|R_\lambda(A)\|_{\mathcal{L}(L^2,L^2)} \leq |\lambda|^{-1}$.  
   
   **Hint:** Write $R_\lambda(A)$ as the composition of the operator in $\mathcal{L}(L^2(S^1), H^2(S^1))$ in (1) with the inclusion of $H^2(S^1)$ into $L^2(S^1)$, using Problem 5 on Problem Set 7.

2. Show that if $\lambda \in \mathbb{C}$, $\lambda \neq n^2$, $n \in \mathbb{Z}$, then $R_\lambda(A)$ is compact as an operator in $\mathcal{L}(L^2(S^1))$.

**Hint:** Write $R_\lambda(A)$ as the composition of the operator in $\mathcal{L}(L^2(S^1), H^2(S^1))$ in (1) with the inclusion of $H^2(S^1)$ into $L^2(S^1)$, using Problem 5 on Problem Set 7.

3. Now suppose $V \in C(S^1)$ is real valued so the corresponding multiplication operator, $M_V \in \mathcal{L}(L^2(S^1))$ is self-adjoint, and let $L = A + M_V \in \mathcal{L}(H^2(S^1), L^2(S^1))$. Fix $\mu > \|M_V\|_{\mathcal{L}(L^2)} = \sup |V|$. Show that $L + \mu I \in \mathcal{L}(H^2(S^1), L^2(S^1))$ is invertible, and its inverse is compact and self-adjoint as an element of $\mathcal{L}(L^2(S^1))$.

**Hint:** Write $L + \mu I = (I + M_V(A + \mu I)^{-1})(A + \mu I)$.

4. Let $\{e_n : n \in \mathbb{N}\}$ be a complete orthonormal set of eigenvectors for the operator $(L + \mu I)^{-1}$. Show that $\{e_n : n \in \mathbb{N}\}$ is also a complete orthonormal set of eigenvectors for $L$, and express the eigenvalues of $L$ in terms of those of $(L + \mu I)^{-1}$. Use this to conclude that the eigenvalues $\lambda_n$ of $L$ tend to $+\infty$. 
