Problem 1. (10 points) Do Problem 4.7. (cf. Problem 4.6.)

Problem 2. (10 points) Do Problem 4.9.

Problem 3. (20 points) Let $V$ be an inner product space. Prove that the inner product can be (uniquely!) extended to the completion, $\tilde{V}$, by continuity (i.e. so that the extension is continuous).

Problem 4. (20 points) Show that elements of $L^1([0,1])$ and $L^2([0,1])$ may be multiplied by continuous functions, i.e. the multiplication operator by $a \in C([0,1])$, $M_a : C([0,1]) \to C([0,1])$, given by $(M_a f)(x) = a(x) f(x)$ for $f \in C([0,1])$, extends uniquely to a continuous linear map $L^1([0,1]) \to L^1([0,1])$, and $\|M_a u\|_{L^1([0,1])} \leq \sup |a| \|u\|_{L^1([0,1])}$, and the analogous statement also for $L^1$ replaced by $L^2$.

Problem 5. (20 points) Let $A > 0$. Suppose that $a_j \in C([0,1])$ and $\sup |a_j| \leq A$ for all $j$ and for all $x_0 > 0$, suppose that on $[x_0, 1]$ $a_j \to 1$ uniformly. Consider the operator $M_{a_j}$ from Problem 4 on $L^1([0,1])$ and on $L^2([0,1])$, respectively. Show that for all $f \in L^1([0,1])$, resp. for all $f \in L^2([0,1])$, $M_{a_j} f \to f$ in $L^1([0,1])$, resp. $L^2([0,1])$.

Hint: Show the conclusion first for $f \in C([0,1])$.

Note: This is a special case of the dominated convergence theorem in measure theory, but you should do it directly using our completion definition.