MATH 174A: PROBLEM SET 8

Suggested Solution

Problem 1. (*Taylor 3.3.15*) Using Exercise 14, prove the Weierstrass approximation theorem: Any $f \in C([a,b])$ is a uniform limit of polynomials. (Hint: Extend f to u as above, approximate u by $p_\epsilon * u$, and expand this in a power series.)

Solution: As indicated in the hint, we extend f to a function $u \in C_c^0(\mathbb{R})$. By the previous exercise, $p_\epsilon * u \to u$ uniformly as $\epsilon \to 0$ on the whole real line. Suppose $[a,b]$ is contained in the ball of radius R centered at origin. Then the power series of the analytic function $p_\epsilon * u$ (since p_ϵ is analytic on $\mathbb{R}$) converges uniformly on $[a,b]$. Take the truncated power series for each $p_{1/n}$, and call it P_n, which is a polynomial with $\|P_n - p_{1/n}\|_\infty < \frac{1}{n}$ on $[a,b]$. Then the sequence of polynomials P_n converges uniformly to u on $[a,b]$, where $u = f$. So we are done.

Problem 2. (*Taylor 3.3.16.*) Suppose $f \in \mathcal{S}(\mathbb{R}^n)$ is supported in $B_R = \{x \in \mathbb{R}^n : |x| \leq R\}$. Show that $\hat{f}(\xi)$ is holomorphic in $\xi \in \mathbb{C}^n$, and satisfies

$$|\hat{f}(\xi + i\eta)| \leq C_N \langle \xi \rangle^{-N} e^{R|\eta|}, \qquad \xi, \eta \in \mathbb{R}^n.$$

Solution: By definition and the fact that f is supported on B_R,

$$\hat{f}(\xi + i\eta) = c \int_{B_R} f(x) e^{-ix\cdot\xi + x\cdot\eta} dx,$$

where c is a constant. Since B_R is compact and the integrand is an entire function in $\xi + i\eta$, so $\hat{f}$ is holomorphic. Using Cauchy-Schwarz,

$$\begin{aligned} |\hat{f}(\xi + i\eta)| &\leq c \left(\int_{B_R} f(x)^2 e^{-2ix\cdot\xi} dx \right)^{1/2} \left(\int_{B_R} e^{2x\cdot\eta} dx \right)^{1/2} \\ &\leq c |\hat{f^2}(2\xi)|^{1/2} e^{R|\eta|} \\ &\leq C_N \langle \xi \rangle^{-N} e^{R|\eta|} \end{aligned}$$

where we have used the fact that $f^2 \in \mathcal{S}(\mathbb{R}^n)$.

Problem 3. (*Taylor 3.4.1.*) Define $M_f u$ by $\langle v, M_f u \rangle = \langle fv, u \rangle$, for $v \in \mathcal{S}(\mathbb{R}^n)$, $u \in \mathcal{S}'(\mathbb{R}^n)$. $M_f u$ is also denoted by fu. Show that $M_f : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ continuously, provided $f \in C^\infty(\mathbb{R}^n)$ and each derivative is polynomial bounded, that is, $|D^\alpha f(x)| \leq C_\alpha \langle x \rangle^{N(\alpha)}$.

Solution: If f is C^∞ and polynomial bounded, then $fv \in \mathcal{S}(\mathbb{R}^n)$ for all $v \in \mathcal{S}(\mathbb{R}^n)$. Therefore M_f is indeed a map from $\mathcal{S}'(\mathbb{R}^n)$ into inself. To see that this map is continuous, suppose u_n converges in u weakly in $\mathcal{S}'(\mathbb{R}^n)$, then we have $\langle fv, u_n\rangle$ converges to $\langle fv, u\rangle$. In other words, $\langle v, M_f u_n\rangle$ converges to $\langle v, M_f u\rangle$. Hence M_f is continuous.

Problem 4. (*Taylor 3.4.2.*) Show that the identity $\xi^\alpha D_\xi^\beta \mathcal{F}f(\xi) = (-1)^{|\beta|}\mathcal{F}(D^\alpha x^\beta f)(\xi)$ from section 3 continues to hold for $f \in \mathcal{S}'(\mathbb{R}^n)$.

Solution: Just compute it directly:

$$\begin{aligned}\langle u(x), \mathcal{F}(D_x^\alpha x^\beta f)(\xi)\rangle &= \langle \mathcal{F}u(\xi), D_x^\alpha x^\beta f(x)\rangle \\ &= (-1)^{|\alpha|}\langle \xi^\beta D_\xi^\alpha \mathcal{F}u(\xi), f(x)\rangle \\ &= \langle \mathcal{F}(D_x^\beta x^\alpha u)(\xi), f\rangle \\ &= \langle D_x^\beta x^\alpha u(x), \mathcal{F}f(\xi)\rangle \\ &= (-1)^{|\beta|}\langle u(x), \xi^\alpha D_\xi^\beta \mathcal{F}f(\xi)\rangle.\end{aligned}$$

So, we have $\xi^\alpha D_\xi^\beta \mathcal{F}f(\xi) = (-1)^{|\beta|}\mathcal{F}(D^\alpha x^\beta f)(\xi)$.

Problem 5. Suppose that $P(D) = \sum_{|\alpha|\le m} a_\alpha D^\alpha$, $a_\alpha \in \mathbb{C}$, is a constant coefficient differential operator. We say that P is elliptic if the polynomial $P_m(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha$ has no zeros $\xi \in \mathbb{R}^n \setminus \{0\}$.

(1) Show that if P is elliptic then there exists $c > 0$ such that $|P_m(\xi)| \ge c|\xi|^m$, $\xi \in \mathbb{R}^n \setminus \{0\}$.
(2) Show that if P is non-zero on $\mathbb{R}^n$, then it is elliptic, and the PDE $P(D)u = f$, $f \in \mathcal{S}'(\mathbb{R}^n)$ given, has a unique solution $u \in \mathcal{S}'(\mathbb{R}^n)$. (Hint: show that $|P(\xi)| \ge c(1 + |\xi|)^m$ for some $c > 0$.) Show also that if $f \in \mathcal{S}(\mathbb{R}^n)$ then $u \in \mathcal{S}(\mathbb{R}^n)$.
(3) Show that if P is elliptic, $u \in \mathcal{S}'(\mathbb{R}^n)$, $P(D)u = f$, and $f \in \mathcal{S}(\mathbb{R}^n)$ then $u \in C^\infty(\mathbb{R}^n)$. (Hint: Let $\chi \in C_c^\infty(\mathbb{R}^n)$ be a 'bump function': $\chi(\xi) = 1$ if $|\xi| < R$, $\chi(\xi) = 0$ if $|\xi| > 2R$. Choose $R > 0$ appropriately, and write $\mathcal{F}u = \chi\mathcal{F}u + (1-\chi)\mathcal{F}u$.)

It is a little harder, but not hard, to prove *elliptic regularity*, namely that if P is elliptic, $P(D)u = f$, $f \in C^\infty(\mathbb{R}^n)$, then $u \in C^\infty(\mathbb{R}^n)$.

Solution:

(1) Consider the function $f(\xi) = \frac{|P_m(\xi)|}{|\xi|^m}$, where $\xi \in \mathbb{R}^n\setminus\{0\}$, since P_m is a homogeneous degree m polynomial, the function f is invariant under scaling, i.e. $f(\lambda\xi) = f(\xi)$ for all $\lambda \in \mathbb{R}\setminus\{0\}$. Therefore,

$$\inf_{\xi\in\mathbb{R}^n\setminus\{0\}} f(\xi) = \inf_{\xi\in\mathbb{S}^{n-1}} f(\xi).$$

As $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ is compact, the infimum is attained and nonzero because P_m has no zeros on $\mathbb{R}^n \setminus \{0\}$ by ellipticity of P. Let $c > 0$ be the infimum. We have $|P_m(\xi)| \geq c|\xi|^m$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$.

(2) $\boxed{P \text{ is elliptic}}$ Note that $|P_m(\xi)| \approx |P(\xi)|$ as $|\xi|$ is large, so we can find an $R > 0$ such that $|P_m(\xi)| \geq \frac{1}{2}|P(\xi)|$ and $|P(\xi)| \geq \frac{1}{2}|P_m(\xi)|$ for all $|\xi| \geq R$. On the compact set $|\xi| = R$, pick η such that $0 < |P_m(\eta)| \leq |P_m(\xi)|$ for all $|\xi| = R$. Then

$$\frac{|P(\xi)|}{(1+|\xi|)^m} \geq \frac{1}{2}\frac{|P_m(\xi)|}{(1+|\xi|)^m} \geq \frac{1}{2}\left(\frac{|\xi|}{R}\right)^m \frac{|P_m(\eta)|}{(1+|\xi|)^m} \geq \frac{|P_m(\eta)|}{2^{m+1}R} > 0.$$

So $|P(\xi)| \geq c(1+|\xi|)^m$ for some $c > 0$. This inequality shows that P is elliptic. Suppose not, then $P_m(\xi) = 0$ for some ξ, by homogeneity, $P_m(\lambda\xi) = 0$ for all λ. Therefore, when $|\xi|$ is large,

$$c(1+|\xi|)^m \leq |P(\xi)| = |P(\xi) - P_m(\xi)| \leq C|\xi|^{m-1}.$$

So $(1 + 1/|\xi|)^m \leq c'/|\xi|$ for $|\xi|$ large, take $|\xi| \to \infty$, we get $1 \leq 0$, a contradiction. So P is elliptic.

$\boxed{P(D)u = f,\ f \in \mathcal{S}'(\mathbb{R}^n) \text{ given, has a unique solution } u \in \mathcal{S}'(\mathbb{R}^n).}$ Take the Fourier transform $\mathcal{F}$, which is an isomorphism on $\mathcal{S}'(\mathbb{R}^n)$, we get

$$P(\xi)\mathcal{F}u(\xi) = \mathcal{F}(P(D)u)(\xi) = \mathcal{F}f(\xi) \in \mathcal{S}'(\mathbb{R}^n).$$

Since P is nonzero on $\mathbb{R}^n$, we can divide through

$$\mathcal{F}u(\xi) = \frac{\mathcal{F}f(\xi)}{P(\xi)} \in \mathcal{S}'(\mathbb{R}^n).$$

As $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is bijective, take the inverse Fourier transform yields a unique solution $u \in \mathcal{S}'(\mathbb{R}^n)$. The same argument shows that if $f \in \mathcal{S}(\mathbb{R}^n)$, then $u \in \mathcal{S}(\mathbb{R}^n)$.

(3) $\boxed{u \in C^\infty(\mathbb{R}^n)}$ Follow the same argument as in (2),

$$P(\xi)\mathcal{F}u(\xi) = \mathcal{F}f(\xi) \in \mathcal{S}(\mathbb{R}^n).$$

Now, choose an R big enough such that, $|P(\xi)| > 1$ for all $|\xi| \geq R$. Then, take $\chi \in C_c^\infty(\mathbb{R}^n)$ be a 'bump function': $\chi(\xi) = 1$ if $|\xi| < R$, $\chi(\xi) = 0$ if $|\xi| > 2R$. So we can write

$$\mathcal{F}u(\xi) = \chi(\xi)\frac{\mathcal{F}f(\xi)}{P(\xi)} + (1-\chi(\xi))\frac{\mathcal{F}f(\xi)}{P(\xi)}.$$

Note that $\mathcal{F}u \in \mathcal{S}'(\mathbb{R}^n)$ as $u \in \mathcal{S}'(\mathbb{R}^n)$, also $\frac{\mathcal{F}f(\xi)}{P(\xi)} \in \mathcal{S}(\mathbb{R}^n)$ since $f \in \mathcal{S}(\mathbb{R}^n)$ and $|P(\xi)| > 1$ outside the ball of radius R. So $\chi(\xi)\frac{\mathcal{F}f(\xi)}{P(\xi)} \in \mathcal{S}'(\mathbb{R}^n)$ and has compact support, so its inverse Fourier transform is C^∞. Therefore, u is a sum of C^∞ function and a Schwartz function, hence is C^∞.

Problem 6. Suppose that (X, d) is a metric space.

(1) Suppose that $f : [0, \infty) \to [0, \infty)$ is C^1 (continuously differentiable), $f(0) = 0$, $f'(0) > 0$, $f'(x) \geq 0$ for all x, and f' is decreasing (i.e. $x \leq y$ implies $f'(x) \geq f'(y)$). Show that $f \circ d : X \times X \to [0, \infty)$ is a metric on X. (Hint: show that f is increasing, and $f(x + y) \leq f(x) + f(y)$ for all $x, y \in [0, \infty)$.)

(2) Suppose d and d' are metrics on X. One says that the topology generated by d' is weaker than the topology of d if every d'-open set is d-open. If d' and d have the same open sets, they are called equivalent.

Show that the topology generated by d' is weaker than the topology generated by d if and only if given $\epsilon > 0$ and $x \in X$ there is $\delta > 0$ such that

$$d(x, y) < \delta \Rightarrow d'(x, y) < \epsilon.$$

Use this to show that with f as in (1), d and $f \circ d$ generate the same topology.

(3) Conclude that if d is a metric on X, then so is $d' = \frac{d}{1+d}$, and these two metrics generate the same topology. Note that $d'(x, y) < 1$ for all $x, y \in X$.

Solution:

(1) $\boxed{f \text{ is increasing}}$ Since $f'(x) \geq 0$ for all x, f is increasing.

$\boxed{f(x + y) \leq f(x) + f(y) \text{ for all } x, y \in [0, \infty)}$ Let $x \in [0, \infty)$. Consider the function $g(y) = f(x) + f(y) - f(x+y)$. Then $g(0) = 0$ and $g'(y) = f'(y) - f'(x+y) \geq 0$ because $y \leq x + y$ and f' decreasing. Thus g is increasing, so $g(y) \geq 0$ for all $y \geq 0$ and thus f is subadditive.

$\boxed{f \circ d \text{ defines a metric}}$ Clearly, $f \circ d \geq 0$, and if $f(d(x, y)) = 0$, then $d(x, y) = 0$, so $x = y$, so $f \circ d$ is positive definite. $f \circ d$ is clearly symmetric, and if $x, y, z \in X$, then $d(x, z) \leq d(x, y) + d(y, z)$ and so

$$f(d(x, z)) \leq f(d(x, y) + d(y, z)) \leq f(d(x, y)) + f(d(y, z)),$$

so $f \circ d$ obeys the triangle inequality and is thus a metric.

(2) $\boxed{\text{Equivalence of the two statements}}$ Suppose that the topology of d' is weaker than that of d. Then, given $\epsilon > 0$ and $x \in X$, we have that $B_{d'}(x, \epsilon)$ is open in (X, d'), so there is an open ball of radius $\delta > 0$ such that $B_d(x, \delta) \subset B_{d'}(x, \epsilon)$, i.e. $d(x, y) < \delta \Rightarrow d'(x, y) < \epsilon$.
Now suppose that for all $x \in X$ and for all $\epsilon > 0$, there is a $\delta > 0$ such that $d(x, y) < \delta \Rightarrow d'(x, y) < \epsilon$. Let U be open in (X, d'), so for all $x \in U$ there is an $\epsilon > 0$ such that $B_{d'}(x, \epsilon) \subset U$. Then there is a $\delta > 0$ such that $B_d(x, \delta) \subset B_{d'}(x, \epsilon) \subset U$, so U is open in (X, d).

$\boxed{d \text{ and } f \circ d \text{ generate the same topology}}$ Now suppose that $\epsilon > 0$. f is continuous, so there is a $\delta > 0$ such that if $|z| < \delta$, then $f(z) < \epsilon$, i.e. for all $x \in X$, if $d(x, y) < \delta$, $f(d(x, y)) < \epsilon$, so the topology of $f \circ d$ is weaker than that of d. Moreover, $f'(0) > 0$, so the inverse function theorem tells us that f is locally (around 0) invertible with continuous inverse, so for all $\epsilon > 0$, there is a $\delta > 0$ such that if $f(z) < \delta$, then $z < \epsilon$, so if $x \in X$, $f(d(x,y)) < \delta$, then $d(x,y) < \epsilon$. Thus the topology of d is weaker than that of $f \circ d$, so they generate the same topology.

(3) $\boxed{d \text{ and } d' \text{ generate the same topology}}$ Consider $f : [0,\infty) \to \mathbb{R}$ given by $f(x) = \frac{x}{1+x}$. Then $f(x) \geq 0$ for all $x \geq 0$, $f(0) = 0$, and $f'(x) = \frac{1}{(1+x)^2} > 0$, which is decreasing, so f is an increasing subadditive function. Thus $d' = f \circ d$ is a metric by parts (1), and it generates the same topology as d by part (2). Note that $f(x) < 1$ for all $x \geq 0$, so $d'(x,y) < 1$ for all $x, y \in X$.

Problem 7. Let $\rho_1, \rho_2, \ldots,$ be metrics on X with $\rho_j \leq 1$. Let

$$d(x,y) = \sum_{j=1}^{\infty} 2^{-j} \rho_j(x,y). \tag{1}$$

(1) Show that d is a metric on X.
(2) Show that a sequence $\{x_n\}$ converges to some $x \in X$ with respect to d if and only if it converges with respect to ρ_j for every j, i.e. if and only if given j and $\epsilon > 0$ there is N such that $n \geq N$ implies $\rho_j(x_n, x) < \epsilon$.
(3) Now suppose that X is a vector space and each d_j is a translation invariant metric, i.e. $d_j(x+z, y+z) = d_j(x,y)$ for all $x, y, z \in X$. Let ρ_j be translation invariant metrics equivalent to d_j with $\rho_j < 1$. Show that a sequence $\{x_n\}$ is Cauchy with respect to d if and only if it is Cauchy with respect to every d_j.
(4) Now suppose that $X_1, X_2, \ldots$ are vector spaces, $X_1 \supset X_2 \supset \ldots$ and $X = \cap_{k=1}^{\infty} X_k$. Let d_k be translation invariant metrics on X_k, and suppose that the inclusion maps $\iota_k : X_k \to X_{k-1}$ are all continuous. Show that if (X_k, d_k) is complete for every k then (X, d) is complete.
(5) Let $C^\infty(\mathbb{S}^1)$ denote the set of complex valued infinitely differentiable functions on $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$. Let d_k be the metric given by the C^k norm:

$$\|f\|_{C^k} = \sum_{m=0}^{k} \sup\{|f^{(m)}(x)| : \ x \in \mathbb{S}^1\}.$$

Let d be the corresponding metric on $C^\infty(\mathbb{S}^1)$. Show that $C^\infty(\mathbb{S}^1)$ is a complete metric space in which sequences $\{x_n\}$ converge, resp. are Cauchy, if and only if they converge, resp. are Cauchy, in every C^k. (Thus, convergence of a sequence $\{f_n\}$ is just the uniform convergence of all derivatives $\{f_n^{(k)}\}$.)

Solution:

(1) $\boxed{d \text{ is a metric}}$ $d(x,y) \leq \sum_{j=1}^{\infty} 2^{-j} = 1$, so $d : X \times X \to [0,\infty)$. All ρ_j are symmetric, so d is symmetric. If $x,y,z \in X$, then

$$d(x,z) = \sum_{j=1}^{\infty} 2^{-j}\rho_j(x,z) \leq \sum_{j=1}^{\infty} 2^{-j}(\rho_j(x,y) + \rho_j(y,z)) =$$

$$\sum_{j=1}^{\infty} 2^{-j}\rho_j(x,y) + \sum_{j=1}^{\infty} 2^{-j}\rho_j(y,z) = d(x,y) = d(y,z),$$

so d satisfies the triangle inequality. Moreover if $d(x,y) = 0$, then $\rho_j(x,y) = 0$ for each j and hence $x = y$. Thus d is a metric on X.

(2) $\boxed{x_n \to x \text{ with respect to } d \text{ iff } x_n \to x \text{ with respect to } \rho_j \text{ for every } j.}$ If x_n converges to x with respect to d, then $d(x_n,x) \to 0$. Let $j \in \mathbb{N}, \epsilon > 0$, then because $d(x_n,x) \to 0$, there is some N such that for all $n \geq N$, $2^{-j}\epsilon > d(x_n,x)$, but $2^{-j}\rho(x_n,x) \leq d(x_n,x) < 2^{-j}\epsilon$, so $\rho_j(x_n,x) < \epsilon$.
Now suppose that $\rho_j(x_n,x) \to 0$ for all j. Let $\epsilon > 0$ and let K be such that $\sum_{j>K} 2^{-j} < \epsilon/2$. We may find N_i such that $n \geq N_i$ implies $\rho_i(x_n,x) < \epsilon/K$, and, letting $N = \max\{N_1,\ldots,N_K\}$, we have that $n \geq N$ implies

$$d(x_n,x) = \sum_{j=1}^{\infty} 2^{-j}\rho_j(x_n,x) = \sum_{j=1}^{K} 2^{-j}\rho_j(x_n,x) + \sum_{j>K} 2^{-j}\rho_j(x_n,x)$$

$$\leq \sum_{j=1}^{K} 2^{-j}\frac{\epsilon}{K} + \sum_{j>K} 2^{-j} \leq \epsilon,$$

so $x_n \to x$ with respect to d.

(3) The same argument as in (2) shows that $\{x_n\}$ is Cauchy with respect to d if and only if it is Cauchy with respect to all ρ_j, but ρ_j are equivalent to d_j, so this happens if and only if x_n are Cauchy with respect to all d_j.

(4) Let ρ_j be equivalent to d_j and bounded by 1 (e.g. $\rho_j = \frac{d_j}{1+d_j}$). Let $\{x_n\}$ be a Cauchy sequence in (X,d), so by (3) it is Cauchy in (X_k,d_k) for all k. (X_k,d_k) is complete, so there is some $z_k \in X_k$ such that $x_n \to z_k$ in X_k, i.e. $d_k(x_n,z_k) \to 0$. $\iota_k : X_k \to X_{k-1}$ is continuous, so, given any k, we have that $\iota_k(x_n) \to \iota_k(z_k)$, i.e. $d_{k-1}(x_n,z_k) \to 0$. $x_n \to z_{k-1}$ in X_{k-1}, so we must have that $z_k = z_{k-1}$ for all k. Thus $z_k = z \in X$ for all k, and $d_k(x_n,z) \to 0$ for all k, so $d(x_n,z) \to 0$ and thus (X,d) is complete.

(5) Let $\rho_k = \frac{d_k}{1+d_k}$, so then by applying parts (2),(3), and (4), it is enough to show that each inclusion $\iota_k : C^k(\mathbb{S}^1) \to C^{k-1}(\mathbb{S}^1)$ is continuous, but this is clear because

$$\|\iota_k f\|_{C^{k-1}} = \|f\|_{C^{k-1}} = \sum_{m=0}^{k-1} \sup\{|f^{(m)}(x)| : x \in \mathbb{S}^1\} \leq \sum_{m=0}^{k} \sup\{|f^{(m)}(x)| : x \in \mathbb{S}^1\} = \|f\|_{C^k},$$

so this inclusion is bounded and thus continuous.