Problem 1. Suppose $V$ is a finite dimensional vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Its dual $V^*$ is the vector space $\mathcal{L}(V, \mathbb{F})$ of linear maps from $V$ to $\mathbb{F}$. The elements of $V^*$ are called linear functionals on $V$.

(1) Show that $V^*$ is finite dimensional, $\dim V^* = \dim V$, and in fact if $e_1, \ldots, e_n$ is a basis of $V$ then the linear functionals $f_1, \ldots, f_n$ defined by

$$f_j(e_k) = \delta_{jk} = \begin{cases} 1, & j = k; \\ 0, & j \neq k, \end{cases}$$

and extended to $V$ by linearity:

$$f_j\left(\sum_k a_k e_k\right) = \sum_k a_k f_j(e_k) = a_j,$$

are a basis of $V^*$. (Hint: Suppose that $f = \sum_j b_j f_j$ and find the $b_j$'s. Now just define the $b_j$ by the resulting formula, and show that they work.) $\{f_1, \ldots, f_n\}$ is called the basis dual to $\{e_1, \ldots, e_n\}$.

(2) If $V$ is real (i.e. $\mathbb{F} = \mathbb{R}$) and has an inner product, there is a natural map $\iota \in \mathcal{L}(V, V^*)$, namely $\iota(v)(w) = (v, w)$, where $(.,.)$ on the right hand side is the inner product. (There's an analogous map if $\mathbb{F} = \mathbb{C}$, but it is conjugate linear.) Show that $\iota$ is a bijection, hence an isomorphism of vector spaces. Thus, given an inner product $V$ can be identified with $V^*$, but the identification depends on the choice of the inner product.

(3) For $v \in V$, consider the map $j \in \mathcal{L}(V, \mathcal{L}(\mathbb{F}, V))$ given as follows: $j(v) \in \mathcal{L}(\mathbb{F}, V)$ is the map $j(v)a = av$, $a \in \mathbb{F}$. Show that $j$ is a bijection from $V$ to $\mathcal{L}(\mathbb{F}, V)$, hence $V$ and $\mathcal{L}(\mathbb{F}, V)$ are isomorphic.

Solution. Part (1). Define $f_j$ as stated. We just need to show that they form a basis of $V^*$, this automatically implies the first two statements. Let’s begin with linear independence. Suppose $a_1, \ldots, a_n \in \mathbb{F}$ and $a_1 f_1 + \cdots + a_n f_n = 0$. Evaluate this function at $e_i$ gives $a_i = 0$. This holds for all $i = 1, \ldots, n$. So they are linearly independent. To see that they span the whole space $V^*$, pick any $f \in V^*$, we claim that $f = f(e_1)f_1 + \cdots + f(e_n)f_n$. Note that $f(e_1)f_1(e_i) + \cdots + f(e_n)f_n(e_i) = f(e_i)$ for any $i = 1, \ldots, n$. But any two linear functional agreeing on a set of basis must be the same. Thus, we are done.

Part (2). We first prove injectivity. Suppose $\iota(v) = 0$. Then by definition, $(v, w) = 0$ for all $w \in V$. In particular, take $w = v$, we have $(v, v) = \|v\|^2 = 0$, hence $v = 0$. Since $V$ and $V^*$ have the same dimension, we get surjectivity for free. Therefore, $\iota$ is a bijection.
**Part (3).** Again both $V$ and $\mathcal{L}(F, V)$ have dimension $n$, it suffices to prove injectivity. Suppose $j(v) = 0$. That is, $av = 0$ for all $a \in \mathbb{F}$. In particular, when $a = 1$, we get $v = 0$. Hence finishing our proof.

**Problem 2.** If $V, W$ are finite dimensional vector spaces over $\mathbb{R}$, $O \subset V$, and $F : O \to W$ is a $C^1$ map, we have defined its derivative $DF(p)$ at $p \in O$ as an element of $\mathcal{L}(V, W)$.

1. If $V$ is a vector space, $\gamma : I \to V$ a $C^1$ curve with $\gamma(0) = p$, show that $D\gamma(0)$ can be naturally identified with an element $\gamma'(0)$ of $V$.
2. Show that the tangent space $T_pV$ of $V$ at $p$, defined as the set of vectors $v$ in $V$ for which there is a curve $\gamma$ with $\gamma(0) = p$ and $\gamma'(0) = v$ is all of $V$. We define the tangent bundle of $V$ as the disjoint union of the $T_pV$, $p \in V$, i.e. since $T_pV = V$, as $TV = V \times V$.
3. If $O \subset V$, $p \in O$, $f : O \to \mathbb{R}$, then $Df(p) \in \mathcal{L}(V, \mathbb{R}) = V^*$. One usually writes $df(p) = DF(p)$. Show that the cotangent space $T^*_pV$ of $V$ at $p$, defined as the set of elements $\alpha$ of $V^*$ for which there is a $C^1$ function $f$ defined near $p$ with $df(p) = \alpha$, is all of $V^*$. We define the cotangent bundle of $V$ as the disjoint union of the $T^*_pV$, $p \in V$, i.e. as $T^*_pV = V^*$, as $T^*V = V \times V^*$. (Note that $TV$ can be identified with $T^*V$ if one is given an inner product, but the identification depends on the inner product.)
4. Notice that $T^*V$ itself is a vector space: $T^*V = V \oplus V^*$. Write elements of $T^*V$ as $w = (v, \alpha)$. We define a map $\Omega : T^*V \times T^*V \to \mathbb{R}$ by
   $$\Omega((v_1, \alpha_1), (v_2, \alpha_2)) = \alpha_1(v_2) - \alpha_2(v_1).$$
   Show that $\Omega$ is bilinear, i.e.
   $$\Omega(cv_1, w_2) = c\Omega(w_1, w_2) = \Omega(w_1, cw_2), \quad c \in \mathbb{R}, \quad w_1, w_2 \in T^*V,$$
   $$\Omega(w_1 + w_2, w_3) = \Omega(w_1, w_3) + \Omega(w_2, w_3), \quad w_1, w_2, w_3 \in T^*V,$$
   with similar additivity in the second slot, $\Omega$ is antisymmetric, i.e.
   $$\Omega(w_1, w_2) = -\Omega(w_2, w_1), \quad w_1, w_2 \in T^*V,$$
   and is non-degenerate, i.e. for $w_1 \in T^*V$ non-zero, there is $w_2 \in T^*V$ such that $\Omega(w_1, w_2) \neq 0$.
5. Note that $\Omega$ (indeed, any bilinear form on $T^*V \times T^*V$) defines a map $J : T^*V \to (T^*V)^*$ as follows: for $w \in T^*V$, $J(w)w' = \Omega(w', w)$. Show that this map is an isomorphism using that $\Omega$ is non-degenerate.

For each $p \in T^*V$, $T_pT^*V$ can be identified with $T^*V$, hence one obtains a non-degenerate bilinear antisymmetric map $\omega_p$ on $T_pT^*V$. It is called the *symplectic form*.

**Solution.** Part (1). We can identify $D\gamma(0)$ with $\gamma'(0)$ by assigning $D\gamma(0)$ with an element of $V$, namely $D\gamma(0)(1)$. This is a natural choice because
   $$D\gamma(0)(1) = \frac{d}{dt} \gamma(t) \bigg|_{t=0}.$$
Part (2). This is obvious since for any \( v \in V \), we can simply take the curve \( \gamma(t) = p + tv \). So \( T_pV \) is all of \( V \).

Part (3). Let \( \alpha \in V^* \). By the result of problem 1, fixing a basis \( \{e_1, \ldots, e_n\} \) of \( V \), we have a canonical dual basis \( \{f_1, \ldots, f_n\} \) of \( V^* \). Under this basis, write \( \alpha = a_1f_1 + \cdots + a_nf_n \). Define a \( C^1 \) function near \( p \), \( f(x) = a_1f_1(x) + \cdots + a_nf_n(x) \). To see that \( df(p) = \alpha \), we just need to look at their actions on the basis elements \( e_i \). By definition,

\[
df(p)(e_i) = \left. \frac{d}{dt} f(p + te_i) \right|_{t=0} = a_i = \alpha(e_i).
\]

So we are done.

Part (4). We first verify that \( \Omega \) linear in the first slot:

\[
\Omega(c(v_1, \alpha_1), (v_2, \alpha_2)) = \Omega((cv_1, c\alpha_1), (v_2, \alpha_2)) \\
= c\alpha_1(v_2) - \alpha_2(cv_1) \\
= c(\alpha_1(v_2) - \alpha_2(v_1)) \\
= c\Omega((v_1, \alpha_1), (v_2, \alpha_2)).
\]

\[
\Omega((v_1, \alpha_1) + (v_2, \alpha_2), (v_3, \alpha_3)) = \Omega((v_1 + v_2, \alpha_1 + \alpha_2), (v_3, \alpha_3)) \\
= (\alpha_1 + \alpha_2)(v_3) - \alpha_3(v_1 + v_2) \\
= \alpha_1(v_3) - \alpha_3(v_1) + \alpha_2(v_3) - \alpha_3(v_2) \\
= \Omega((v_1, \alpha_1), (v_3, \alpha_3)) + \Omega((v_2, \alpha_2), (v_3, \alpha_3)).
\]

Next, we prove antisymmetry:

\[
\Omega((v_1, \alpha_2), (v_2, \alpha_2)) = \alpha_1(v_2) - \alpha_2(v_1) = -(\alpha_2(v_1) - \alpha_1(v_2)) = \Omega((v_2, \alpha_2), (v_1, \alpha_1)).
\]

These two things together implies linearity in the second slot. We still have to show that it is non-degenerate. Suppose there is a \( w_1 = (v_1, \alpha_1) \in T^*V \) such that \( \Omega(w_1, w) = 0 \) for all \( w \in T^*V \). Take \( w = (0, \alpha) \), we have \( \alpha(v_1) = 0 \) for all \( \alpha \in V^* \). This implies that \( v_1 = 0 \). On the other hand, if we take \( w = (v, 0) \), we get \( \alpha_1(v) = 0 \) for all \( v \in V \), thus \( \alpha_1 = 0 \). In conclusion, we are forced to have \( w_1 = 0 \). This proves that \( \Omega \) is a non-degenerate 2-form on \( T^*V \).

Part (5). Again by dimension argument, we just need to show that \( J \) is one-to-one. Suppose \( J(w) = 0 \). Equivalently, for all \( w' \in T^*V \), \( J(w)(w') = \Omega(w', w) = 0 \). By non-degeneracy of \( \Omega \), we must have \( w = 0 \). This finishes the whole proof.

Problem 3. (Taylor I.3.5). Let \( \mathcal{O} \subset \mathbb{R}^n \) be open, \( p \in \mathcal{O} \), and \( f : \mathcal{O} \to \mathbb{R}^n \) be real analytic, with \( Df(p) \) invertible. Take \( f^{-1} : V \to U \) as in Theorem 3.1. Show that \( f^{-1} \) is real analytic. (Hint: Consider a holomorphic extension \( F : \Omega \to \mathbb{C}^n \) of \( f \), and apply Exercise 3.) (Assume that if \( F : \mathcal{O} \to \mathbb{C}^m \) is holomorphic, where \( \mathcal{O} \subset \mathbb{C}^n \) is open, and \( p \in \mathcal{O} \), then the Taylor series of \( F \) converges in a neighborhood of \( p \). We will prove this in the second half of the course.)
**Solution.** First, extend \( f \) to a holomorphic function \( F : \Omega \to \mathbb{C}^n \). We check that \( DF(p) \) as a linear map between \( \mathbb{R}^{2n} \) is invertible. Note that

\[
DF(p) = \begin{bmatrix} Df(p) & O \\ O & Df(p) \end{bmatrix}.
\]

So \( DF(p) \) is invertible. By inverse function theorem, locally there is a \( C^1 \) inverse function \( F^{-1} \) with \( D(F^{-1})(F(p)) = (DF(p))^{-1} \). Since \( F \) is holomorphic, it satisfies \( JDF(p) = DF(p)J \). This implies that \( JD(F^{-1})(F(p)) = D(F^{-1})(F(p))J \). Hence \( F^{-1} \) is also holomorphic and it has a power series expansion around \( F(p) = f(p) \). Restricting on the real line gives a power series of \( f^{-1} \) locally, therefore \( f^{-1} \) is real analytic.

**Problem 4. (Taylor I.6.1).** Let \( \Omega \) be open in \( \mathbb{R}^{2n} \), identified with \( \mathbb{C}^n \), via \( z = x + iy \). Let \( X : \Omega \to \mathbb{R}^{2n} \) have components \( X = (a_1, \ldots, a_n, b_1, \ldots, b_n) \), where \( a_j(x,y) \) and \( b_j(x,y) \) are real-valued. Denote the solution to \( du/dt = X(u) \), \( u(0) = z \) by \( u(t, z) \). Assume \( f_j(z) = a_j(z) + ib_j(z) \) is holomorphic in \( z \), that is, its derivative commutes with \( J \), acting on \( \mathbb{R}^k = \mathbb{C}^k \) as multiplication by \( i \). Show that, for each \( t \), \( u(t, z) \) is holomorphic in \( z \), that is, \( D_z u(t, z) \) commutes with \( J \). (Hint: Use the linearized equation (6.2) to show that \( K(t) = [W(t), J] \) satisfies the ODE

\[
K' = DX(z)K, \quad K(0) = 0.
\]

**Solution.** Define \( K(t) = [W(t), J] = W(t)J - JW(t) \). Then clearly, \( K(0) = W(0)J - JW(0) = IJ - JI = J - J = 0 \). Also,

\[
\]

Since \( W \) is holomorphic by the assumptions on its coefficients \( a_i, b_i \), \( JDW(z) = DW(z)J \). We have

\[
K'(t) = DX(z)W(t)J - DX(z)JW(t) = DX(z)K(t).
\]

Since \( K(t) \) satisfies such a linear ODE, we know that \( K(t) \) is defined on all \( t \) and by uniqueness of solutions, we have \( K(t) = 0 \) for all \( t \), which means that \( W(t) \) and \( J \) commutes, i.e. \( u(t, z) \) is holomorphic in \( z \).

**Problem 5. (Taylor I.6.2).** If \( \mathcal{O} \subset \mathbb{R}^n \) is open and \( F : \mathcal{O} \to \mathbb{R}^n \) is real analytic, show that the solution \( y(t, x) \) to (6.1) is real analytic in \( x \). (Hint: With \( F = (a_1, \ldots, a_n) \), take holomorphic extensions \( f_j(z) \) of \( a_j(x) \) and use Exercise 1.) Using the trick leading to (6.18), show that \( y(t, x) \) is real analytic jointly in \( (t, x) \).

**Solution.** Write \( F = (F_1, \ldots, F_n) \), by assumption each \( F_i \) is real analytic, we can thus locally extend to a holomorphic function \( f_i \), put \( f_i = a_i + ib_i \). Then \( X = (a_1, \ldots, a_n, b_1, \ldots, b_n) \) defines a vector field on \( \mathbb{R}^{2n} \) satisfying all the hypothesis in problem 4. Therefore the solution \( u(t, z) \) to the ODE \( du/dt = X(u) \), \( u(0) = z \), is holomorphic in \( z \), thus \( y(t, x) \) is real analytic in \( x \).
To show that it is actually real analytic jointly in \((t, x)\), consider the solution \(y(t, \tau, x)\) to the family of equations:
\[
y' = \tau F(y) , y(0) = x.
\]
In this case, by uniqueness theorem, \(y(t, \tau, x) = y(\tau t, 1, x)\). Hence \(dy/d\tau = tF(y), y(\tau = 0) = x\). Using the same trick in (6.18) and apply the result we have just proved, \(y(t, x)\) is real analytic jointly in \((t, x)\).

**Problem 6.** (Taylor I.7.1). Suppose \(h(x, y)\) is homogeneous of degree 0, that is, \(h(rx, ry) = h(x, y)\), so \(h(x, y) = k(x/y)\). Show that the ODE
\[
\frac{dy}{dx} = h(x, y)
\]
is changed to a separable ODE for \(u = u(x)\), if \(u = y/x\).

**Solution.** Do the substitution \(u = u(x) = y/x\). We have \(y = ux\),
\[
\frac{dy}{dx} = x\frac{du}{dx} + u = h(x, y) = k(u)
\]
Rearranging, we get a separable ODE
\[
\frac{1}{k(u) - u} \frac{du}{dx} = \frac{1}{x}.
\]

**Problem 7.** (Taylor I.7.2). Using Exercise 1, discuss constructing the integral curves of a vector field
\[
X = f(x, y) \frac{\partial}{\partial x} + g(x, y) \frac{\partial}{\partial y}
\]
when \(f(x, y)\) and \(g(x, y)\) are homogeneous of degree \(a\), that is,
\[
f(rx, ry) = r^a f(x, y)\text{ for } r > 0,
\]
and similarly for \(g\).

**Solution.** We have to solve the ODE
\[
\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)} = h(x, y).
\]
Note that \(g\) and \(f\) are homogeneous of the same degree, \(h\) is homogeneous of degree 0. By the result of problem 6, we get an ODE for \(u(x) = y/x\):
\[
\frac{1}{k(u) - u} \frac{du}{dx} = \frac{1}{x} , \text{where } k(u) = \frac{g(1, u)}{f(1, u)}.
\]

**Problem 8.** (Taylor I.7.3). Describe the integral curves of
\[
(x^2 + y^2) \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}.
\]
Solution. Apply the method of problem 7, we have to solve
\[
\frac{1}{1+u^2} \frac{du}{dx} = \frac{1}{x} \quad \quad \text{and} \quad \quad -\frac{1+u^2}{u^3} du = \frac{1}{x} dx
\]
Integrating both sides,
\[
\frac{1}{2u^2} - \ln u + C = \ln x,
\]
rearranging gives
\[
y = Ce^{\frac{x^2}{2u^2}}.
\]

Problem 9. (Taylor I.7.9). Suppose all the eigenvalues of \(A\) have negative real part. Construct a quadratic polynomial \(Q : \mathbb{R}^n \to [0, \infty)\), such that \(Q(0) = 0\), \((\partial^2 Q/\partial x_j \partial x_k)\) is positive-definite, and for any integral curve \(x(t)\) of \(X\) as in (7.25),
\[
\frac{d}{dt} Q(x(t)) < 0 \quad \text{if} \quad t \geq 0,
\]
provided \(x(0) = x_0 (\neq 0)\) is close enough to 0. Deduce that for small enough \(C\), if \(\|x_0\| \leq C\), then \(x(t)\) exists for all \(t \geq 0\) and \(x(t) \to 0\) as \(t \to \infty\). (Hint: Take \(Q(x) = \langle x, x \rangle\), using Exercise 10 below.) Assuming that \(A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)\) has \(n\) real linearly independent eigenvectors, and each eigenvalue is negative. (Hint: show that \(A\) is self-adjoint with respect to some inner product \(\langle \cdot, \cdot \rangle\) on \(\mathbb{R}^n\), i.e. \(\langle Au, v \rangle = \langle u, Av \rangle\) for \(u, v \in \mathbb{R}^n\).)

Solution. By assumption, \(A\) has \(n\) real linearly independent eigenvectors \(v_1, \ldots, v_n\) with \(Av_i = -\lambda_i v_i\) where \(-\lambda_i < 0\) are the eigenvalues of \(A\). Define an inner product on \(\mathbb{R}^n\) for setting \(v_1, \ldots, v_n\) to be an orthonormal basis of this inner product \(\langle \cdot, \cdot \rangle\). Take \(Q(x) = \langle x, x \rangle\), then by the properties of inner product, it is clear that \(Q(0) = 0\) and \((\partial^2 Q/\partial x_j \partial x_k)\) is positive-definite. Let \(x(t)\) be an integral curve of \(X\) with \(x(0) = x_0\) close to zero enough such that \(X(x) \approx Ax\). Then
\[
\frac{d}{dt} Q(x(t)) = 2\langle x(t), x'(t) \rangle = 2\langle x(t), X(x(t)) \rangle = 2\langle x(t), Ax(t) \rangle.
\]
If we write \(x = x_1 v_1 + \cdots + x_n v_n\), then
\[
\frac{d}{dt} Q(x(t)) = -x_1^2 \lambda_1 - \cdots - x_n^2 \lambda_n < 0.
\]
For the last statement, simply note that \(Q(x(t)) = \|x(t)\|^2\) is a strictly decreasing function in \(t\) and the local behaviour of \(X\) around 0 forces \(x(t) \to 0\) as \(t \to \infty\).