Problem 1. (Cf. Taylor I.1.3.) Let $M_{n \times n}(\mathbb{C})$ denote the set of $n \times n$ complex matrices. Suppose $A \in M_{n \times n}(\mathbb{C})$ is invertible. Using

$$\det(A + tB) = (\det A) \det(I + tA^{-1}B)$$

show that

$$D \det(A)B = (\det A) \text{Tr}(A^{-1}B).$$

(Hint: you have already shown $D \det(I)B = \text{Tr} B$.)

Note: this shows that $\text{SL}_n(\mathbb{C})$ defined as the set of matrices $A \in M_{n \times n}$ with $\det A = 1$ is a $C^\infty$, indeed holomorphic, (hyper)surface in $M_{n \times n} = \mathbb{C}^{n^2}$: take $B = A$ to conclude that $D \det(A)$ is surjective. (The same calculation using $M_{n \times n}(\mathbb{R})$ shows that $\text{SL}_n(\mathbb{R})$ is real analytic.)

Solution.

$$D \det(A)B = \frac{d}{dt} \det(A + tB)\bigg|_{t=0} = (\det(A)) \frac{d}{dt} \det(I + tA^{-1}B)\bigg|_{t=0} = (\det(A))D \det(I)(A^{-1}B) = (\det(A)) \text{Tr}(A^{-1}B).$$

Problem 2. Let $O_n(\mathbb{R})$ denote the set of matrices $A \in M_{n \times n}(\mathbb{R})$ with the property that $AA^t = I$; here $A^t$ denotes the transpose of $A$ (i.e. the $ij$ entry of $A^t$ is the $ji$ entry of $A$). Let $S_{n \times n}(\mathbb{R})$ denote the set of symmetric matrices, i.e. matrices $A$ such that $A^t = A$. Note that $S_{n \times n}$ can be identified with $\mathbb{R}^{\frac{n(n+1)}{2}}$ as for symmetric matrices $A$, the below diagonal entries are determined by the above diagonal entries.

Consider the map $F : M_{n \times n} \to S_{n \times n}$ given by $F(A) = AA^t$. Show that

$$(DF)(A)B = AB^t + BA^t,$$

and show that for $A \in O_n(\mathbb{R})$, $DF(A) : M_{n \times n} \to S_{n \times n}$ is surjective.

Use this to show that $O_n(\mathbb{R})$ is a compact surface in $M_{n \times n}$ of dimension $\frac{n(n-1)}{2}$. $O_n(\mathbb{R})$ is called the orthogonal group on $\mathbb{R}^n$. 

Solution.

\[(DF)(A)(B) = \frac{d}{ds} F(A + sB) \bigg|_{s=0} = \frac{d}{ds} (A + sB)(A + sB)^t \bigg|_{s=0} = \frac{d}{ds} (AA^t + sAB^t + sBA^t + s^2 BB^t) \bigg|_{s=0} = AB^t + BA^t.\]

To show that for any \(A \in O_n(\mathbb{R})\), \(DF(A) : M_{n \times n} \to S_{n \times n}\) is surjective, pick an arbitrary \(C \in S_{n \times n}\) (i.e. \(C^t = C\)), take \(B = \frac{1}{2} CA\), then

\[(DF)(A)(B) = AB^t + BA^t = \frac{1}{2} (AA^t C^t + CAA^t) = \frac{1}{2} (C^t + C) = C,\]

where we have used the fact that \(AA^t = I\) as \(A \in O_n(\mathbb{R})\).

For the last statement, first observe that \(O_n(\mathbb{R}) = \{A \in M_{n \times n} : AA^t = I\} = F^{-1}(I)\). \(F\) is clearly continuous map, hence \(O_n(\mathbb{R})\) is closed as the preimage of a closed set. Also, \(O_n(\mathbb{R})\) can be interpreted as those matrices such that its columns (or rows) form an orthonormal basis of \(\mathbb{R}^n\), thus the absolute values of its entries are all no bigger than 1. Hence \(O_n(\mathbb{R})\) is a closed and bounded set in \(\mathbb{R}^{n^2}\), which is compact by Heine-Borel. Since \(DF(A)\) is surjective at each \(A \in O_n(\mathbb{R})\), by implicit function theorem, \(O_n(\mathbb{R})\) has dimension equals to \(\dim(M_{n \times n}) - \dim(S_{n \times n}) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}\).

**Problem 3.** Suppose that \(M\) is a smooth \(k\)-dimensional surface in \(\mathbb{R}^n\). Show that for each \(p \in M\), the set of vectors tangent to \(M\) at \(p\) form a \(k\)-dimensional linear subspace of \(\mathbb{R}^n\). (Hint: Use the straightening out of the previous problem set.) We denote this by \(T_pM\).

Show also that if on a neighborhood \(O\) of \(p\) in \(\mathbb{R}^n\), \(M\) is defined by \(\Phi = 0\), then \(T_pM\) is the nullspace of \(D\Phi(p) : \mathbb{R}^n \to \mathbb{R}^{n-k}\).

Use this to conclude that the disjoint union of tangent spaces \(T_pM, p \in M\), is a \(2k\)-dimensional surface in \(\mathbb{R}^{2n}\): consider the set

\[TM = \{(p, v) \in \mathbb{R}^{2n} : p \in M, v \in T_pM\},\]

and show that the map

\[F : O \times \mathbb{R}^n \to \mathbb{R}^{2(n-k)}, \quad F(x, v) = (\Phi(x), D\Phi(x)v)\]

defines \(TM\) on \(O \times \mathbb{R}^n\). (That is, \(TM\) is given by \(F = 0\), and \(DF\) is surjective on \(TM\).) \(TM\) is called the tangent bundle of \(M\).

Note: Let \((\cdot, \cdot)\) denote the standard inner product on \(\mathbb{R}^n\). Every \(v \in \mathbb{R}^n\) defines a linear map \(\iota(v) : \mathbb{R}^n \to \mathbb{R}\) by \(\iota(v)w = (v, w)\). Conversely, for every linear map \(A : \mathbb{R}^n \to \mathbb{R}\) there
is a vector $v \in \mathbb{R}^n$ such that for all $w \in \mathbb{R}^n$, $(v, w) = Aw$, i.e. $\iota$ is surjective. ($\iota$ is also injective.)

Now, if $\Phi = (\Phi_1, \ldots, \Phi_{n-k})$, then $D\Phi_j(p)$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}$, $\nabla\Phi_j(p)$ denotes the image of $D\Phi_j(p)$ under $\iota^{-1}$; it is of course just $(\partial_1 \Phi_j, \ldots, \partial_n \Phi_j)$.

Thus, we can reinterpret the result above: $T_pM$ is the orthocomplement of the span of $\nabla\Phi_1(p), \ldots, \nabla\Phi_{n-k}(p)$.

**Solution.** Using the straightening out technique in our previous problem set, we know that around a small neighborhood of $p$, the surface can be parametrized by $m$ coordinates $x_1, \ldots, x_m$ by the map $g(x_1, \ldots, x_m) = (x_1, \ldots, x_m, f_1(x_1, \ldots, x_m), \ldots, f_{n-k}(x_1, \ldots, x_m))$.

Then, it is easy to see that the vectors $\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_k}$ forms a basis for the tangent space of $M$ at $p$.

Assume that around a small neighborhood of $p$, $U, M$ is defined by the equation $\Phi = 0$. Then any curve lying on $O$ through $p$ is constantly zero under the map, this shows that the differential map $D\Phi$ vanishes on every tangent vectors of $M$ at $p$. Since $D\Phi$ is surjective as a defining function of $M$, be considering dimensions, we conclude that $T_pM$ is exactly the kernel of $D\Phi$.

Define $F : O \times \mathbb{R}^n \to \mathbb{R}^{2(n-k)}$ as $F(x, v) = (\Phi(x), D\Phi(x)v)$. Then $F(x, v) = 0$ if and only if $\Phi(x) = 0$ and $D\Phi(x)(v) = 0$ if and only if $x \in O$ and $v \in T_pM$. Hence, $TM$ is given by $F = 0$. Note that

$$DF = \begin{bmatrix} D\Phi & O \\ \ast & D\Phi \end{bmatrix},$$

which is surjective since $D\Phi$ is surjective.

**Problem 4.** (Taylor I.4.9.) Given $X \in M_n$, define $\text{ad}X \in \text{End}(M_n)$, that is, $\text{ad}X : M_n \to M_n$ by $\text{ad}X(Y) = XY - YX$. Show that

$$e^{-tX}Ye^{tX} = e^{-t\text{ad}X}Y.$$  

(Hint: If $V(t)$ denotes either side, show that $dV/dt = -(\text{ad}X)(V), V(0) = Y$.)

**Solution.** Follow the hint, it is clear that both sides satisfy the initial condition $V(0) = Y$. Moreover,

$$\frac{d}{dt}e^{-tX}Ye^{tX} = -Xe^{-tX}Ye^{tX} + e^{-tX}Ye^{tX} = -X(e^{-tX}Ye^{tX}) + (e^{-tX}Ye^{tX})X,$$

$$\frac{d}{dt}e^{-t\text{ad}X}Y = -\text{ad}Xe^{-t\text{ad}X}Y.$$

Hence both sides satisfies the ODE $dV/dt = -(\text{ad}X)(V), V(0) = Y$, by uniqueness theorem for this kind of ODE with constant coefficients, we know that both sides agree on all $t \in \mathbb{R}$. 
Problem 5. (Taylor I.5.1.) Let $A(t)$ and $X(t)$ be $n \times n$ matrices satisfying
\[ \frac{dX}{dt} = A(t)X. \]
We form the Wronskian $W(t) = \det X(t)$. Show that $W$ satisfies the ODE
\[ \frac{dW}{dt} = a(t)W, \quad a(t) = \text{Tr} A(t). \]
(Use the alternative hint: Write $X(t + h) = e^{hA(t)}X(t) + O(h^2)$ and use Exercise 3 of section 4 to write $\det e^{hA(t)} = e^{hA(t)}$, hence $W(t + h) = e^{hA(t)}W(t) + O(h^2)$.)

Solution. For $h$ small enough, we have from the Taylor expansion
\[ X(t + h) = X(t) + h \frac{dX}{dt}(t) + O(h^2) = X(t) + hA(t)X + O(h^2) = e^{hA(t)}X + O(h^2). \]
Taking determinant on both sides, using Exercise 3 or section 4,
\[ W(t+h) = \det(e^{hA(t)}) \det(X(t)) + O(h^2) = e^{\text{Tr} hA(t)} \det(X(t)) + O(h^2) = e^{hA(t)}W(t) + O(h^2), \]
thus $dW/dt = a(t)W(t)$.

Problem 6. (Taylor I.5.2.) Let $u(t) = \|y(t)\|^2$, for a solution $y$ to (5.1). Show that
\[ u' \leq M(t)u(t), \]
provided $\|A(t)\| \leq M(t)/2$. Such a differential inequality implies the integral inequality
\[ u(t) \leq A + \int_0^t M(s)u(s) \, ds, \quad t \geq 0, \]
with $A = u(0)$. The following is a Gronwall inequality; namely, if (5.17) holds for a real-valued function $u$, then provided $M(s) \geq 0$, we have, for $t \geq 0$,
\[ u(t) \leq Ae^{N(t)}, \quad N(t) = \int_0^t M(s)ds. \]
Prove this. Note that the quantity dominating $u(t)$ in (5.18) is equal to $U$, solving $U(0) = A$, $dU/dt = M(t)U(t)$.

Solution. Write $u(t) = (y(t), y(t))$ where $(\cdot, \cdot)$ denotes the standard inner product in $\mathbb{R}^n$, we have
\[ u'(t) = 2(y(t), y'(t)) \leq 2\|y(t)\|\|y'(t)\| = 2\|y(t)\|\|A(t)y(t)\| \leq M(t)u(t). \]
Direct integration on both sides give the integral form of this inequality. Let
\[ N(t) = \int_0^t M(s)ds. \]
Rearranging (5.17) gives (when $A = 0$, there is nothing to prove, so we can assume $A > 0$)
\[ \frac{u(t)}{A + \int_0^t M(s)u(s)ds} \leq 1. \]
Multiplying both sides by $M(t)$ (we use the non-negativity of $M$ here) then integrate

$$\ln \left( A + \int_0^t M(s)u(s)ds \right) - \ln(A) \leq N(t),$$

Taking exponential and apply (5.17) again

$$u(t) \leq A + \int_0^t M(s)u(s)ds \leq Ae^{N(t)}.$$