Problem 1. Recall that a map $F$ between complex vector spaces is called differentiable (in the complex sense) at $x$ if there is a complex linear map $L$ such that $F(x + y) = F(x) + Ly + R(x,y)$ with $\lim_{\|y\| \to 0} \frac{\|R(x,y)\|}{\|y\|} = 0$. If $F$ is differentiable as a complex map, with $DF$ continuous, one calls $F$ holomorphic, or complex analytic.

Identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by writing $z \in \mathbb{C}^n$ as $z = x + iy$, $x, y \in \mathbb{R}^n$. Multiplication by $i$ on $\mathbb{C}^n$ becomes multiplication by a matrix $J_n$ on $\mathbb{R}^{2n}$, $J_n = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$, where $I$ is the identity operator on $\mathbb{R}^n$.

Show that for $O \subset \mathbb{C}^n$ open, $F : O \to \mathbb{C}^m$ is holomorphic if and only if $F$ is $C^1$ as a real map (i.e. identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, $\mathbb{C}^m$ with $\mathbb{R}^{2m}$), and $DF \in \mathcal{L}(\mathbb{R}^{2n}, \mathbb{R}^{2m})$, regarded as a real linear map, satisfies $J_mDF(z) = DF(z)J_n$

for all $z \in O$.

Write out $DF(z)$ as a block matrix corresponding to $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$, and similarly with $\mathbb{R}^{2m}$, writing $F = u + iv$, $u, v : O \to \mathbb{R}^m$, and deduce the Cauchy-Riemann equations $D_xu = D_yv, \ D_yu = -D_xv$

hold if and only if $F$ is holomorphic.

Solution. Let $O \subset \mathbb{C}^n$ be an open set.

First, assume that $F : O \to \mathbb{C}^m$ is holomorphic. Let $z = x + iy \in O$. There is a complex linear map $DF(z) : \mathbb{C}^n \to \mathbb{C}^m$ such that $\lim_{\|w\| \to 0} \frac{\|F(z+w) - F(z) - DF(z)(w)\|}{\|w\|} = 0$. Moreover, the linear maps $DF$ depends continuously on $z$, namely, if we pick a basis on $\mathbb{C}^n$ and write $DF$ as a matrix with respect to the basis, then all the entries are continuous functions of $z \in O$. $DF(z)$ is complex linear, in particular, can be regarded as a real linear map from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2m}$. Since all vector spaces are finite dimensional here, $DF(z)$ is automatically continuous, i.e. bounded. In other words, $DF(z) \in \mathcal{L}(\mathbb{R}^{2n}, \mathbb{R}^{2m})$ for all $z \in O$. In addition, it means that $F$ is differentiable in the real sense since $DF$ is also real linear. By the corresponding result for real differentiable maps, we know that $F$ is $C^1$ as a real map. It remains to verify that for any $z \in O$, we have $J_mDF(z) = DF(z)J_n$. But this is obvious since the action of $J_m$ on a vector is multiplication by $i$ and $DF(z)$ is a complex linear map.
Next, we assume that $F$ is $C^1$ as a real map, $DF(z) \in L(\mathbb{R}^{2n}, \mathbb{R}^{2m})$ and $J_mDF(z) = DF(z)J_n$ for all $z \in O$. We want to show that $F$ is holomorphic. Since $J_mDF(z) = DF(z)J_n$, $DF(z)$ is actually a complex linear map from $\mathbb{C}^n$ to $\mathbb{C}^m$. Therefore, $F$ is complex differentiable at every $z \in O$. Also, $F$ is $C^1$ implies that $DF(z)$ is continuous as a function of $z$. All these together means that $F$ is holomorphic on $O$.

A direct calculation shows that

$$DF(z) = \begin{bmatrix} D_xu(z) & D_xv(z) \\ D_yu(z) & D_yv(z) \end{bmatrix}.$$  

Therefore, the condition $J_mDF(z) = DF(z)J_n$ translates into

$$\begin{bmatrix} -D_yu & -D_yv \\ D_xu & D_xv \end{bmatrix} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} D_xu & D_xv \\ D_yu & D_yv \end{bmatrix} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} D_xv & -D_xu \\ D_yv & -D_yu \end{bmatrix}.$$  

Comparing their entries gives the Cauchy Riemann Equations:

$$D_xu = D_yv, \quad D_yu = -D_xv.$$  

**Problem 2.** Show that all norms on a finite dimensional vector space $V$ are equivalent, i.e. if $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on $V$ then there is a constant $C > 0$ such that $\|v\|_1 \leq C\|v\|_2$ and $\|v\|_2 \leq C\|v\|_1$ for all $v \in V$. (Hint: You may assume $V$ is real as every complex vector space is also a real vector space. Use a basis to reduce the question to norms on $\mathbb{R}^n$. Now use the fact that the unit sphere in $\mathbb{R}^n$ is compact in the Euclidean topology, and show that all norms on $\mathbb{R}^n$ are equivalent to the Euclidean norm.)

**Solution.** Following the hints, we just need to show that any norm on $\mathbb{R}^n$ is equivalent to the Euclidean norm. Let $B$ denote the unit sphere in the Euclidean topology which is compact. Let $\|\cdot\|'$ be a norm on $\mathbb{R}^n$ and $\|\cdot\|$ be the usual Euclidean norm. Since $\|\cdot\|': \mathbb{R}^n \to \mathbb{R}$ is continuous, by extreme value theorem, $\|\cdot\|'$ attains its maximum, say $C_1 > 0$, and minimum, say $C_2 > 0$, on $B$. Thus $C_2\|v\| \leq \|v\|' \leq C_1\|v\|$ for all $v \in \mathbb{R}^n$. Hence the two norms are equivalent.

**Problem 3.** (*Taylor Ex.I.2.3*) Let $M$ be a compact, smooth surface in $\mathbb{R}^n$. Suppose $F: \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map (vector field) such that, for each $x \in M$, $F(x)$ is tangent to $M$, that is, the line $\gamma_x(t) = x + tF(x)$ is tangent to $M$ at $x$, at $t = 0$. Show that if $p \in M$, then the initial-value problem

$$y' = F(y), \quad y(0) = p$$

has a solution for all $t \in \mathbb{R}$, and $y(t) \in M$ for all $t$. (Hint: Locally, straighten out $M$ to be a linear subspace of $\mathbb{R}^n$, to which $F$ is tangent. Use uniqueness. Material in section 3 will help do this local straightening.) Reconsider this problem after reading section 7.
For this problem, take the following extrinsic definition of a smooth surface in $\mathbb{R}^n$: A smooth $k$-dimensional surface $M \ (0 \leq k \leq n)$ is a subset of $\mathbb{R}^n$, with the relative topology, such that for each $p \in M$ there is a neighborhood $O$ of $p$ in $\mathbb{R}^n$ and a smooth map: $\Phi : O \to \mathbb{R}^{n-k}$ such that

$$M \cap O = \Phi^{-1}(\{0\}) = \{x \in O : \Phi(x) = 0\}$$

and $D\Phi(p) : \mathbb{R}^n \to \mathbb{R}^{n-k}$ is a surjective linear map.

A smooth hypersurface is a smooth $n-1$-dimensional surface in $M$, in which case $\Phi$ is simply a real valued function, and the assumption is that $D\Phi(p) \neq 0$.

The meaning of smooth depends on the context, it usually means either $C^1$ or $C^\infty$. ($C^1$ suffices here.)

Also use the following definition of tangency. First, a curve in $M$ is a smooth map $\gamma : I \to \mathbb{R}^n$ and $\gamma(t) \in M$ for $t \in I$; here $I$ is an interval. A vector $V \in \mathbb{R}^n$ is tangent to $M$ at $p \in M$ if there is a curve $\gamma : I \to M$, $I$ a neighborhood of 0, $\gamma(0) = p$, such that $\gamma'(0) = V$. A vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ is tangent to $M$ if for all $x \in M$, $F(x)$ is tangent to $M$ at $x$.

In order to do the ‘straightening out’ of the hint, use the implicit function theorem by breaking up the standard coordinate functions on $\mathbb{R}^n$ into two groups (which you may then rearrange), $(x, z), x \in \mathbb{R}^k, z \in \mathbb{R}^{n-k}$, such that $D_z\Phi(p)$ is invertible, so on a neighborhood $O' \subset O$ of $p$, $M \cap O' = \{(x, f(x)) : x \in U\}, \ U \subset \mathbb{R}^k$ open, $f$ smooth.

Now define a vector field $\tilde{F} : U \to \mathbb{R}^k$ such that solutions of $x'(t) = \tilde{F}(x(t))$ lift to solve the original ODE, i.e. letting $y(t) = (x(t), f(x(t))), y'(t) = F(y(t))$.

**Solution.** We assume that the surface $M$ is of dimension $0 \leq k \leq n$.

Note that we want $y(0) = p \in M$. We first show that the differential equation is solvable locally around $p$. By the very definition of a smooth $k$-dimensional surface, there is a neighborhood $O$ of $p$ in $\mathbb{R}^n$ and a smooth map $\Phi : O \to \mathbb{R}^{n-k}$ such that

$$M \cap O = \Phi^{-1}(\{0\}) = \{x \in O : \Phi(x) = 0\}$$

and $D\Phi(p) : \mathbb{R}^n \to \mathbb{R}^{n-k}$ is a surjective linear map.

This means that $D\Phi(p)$ is full rank and thus there is an $(n - k) \times (n - k)$ invertible submatrix. WLOG, assume it is formed by the last $n - k$ rows(variables) of $\mathbb{R}^n$. By implicit function theorem, you can break up the standard coordinate functions on $\mathbb{R}^n$ into two groups (which you may then rearrange), $(x, z), x \in \mathbb{R}^k, z \in \mathbb{R}^{n-k}$, such that $D_z\Phi(p)$ is invertible, so on a neighborhood $O' \subset O$ of $p$, $M \cap O' = \{(x, f(x)) : x \in U\}, \ U \subset \mathbb{R}^k$ open, $f$ smooth. In more technical terms, the map $\psi : U \to \mathbb{R}^n$, $\psi(x) = (x, f(x))$ gives a local coordinate system of the surface around $p$. 
Next, we will solve the ODE locally around $p$ using this coordinate system. Consider the following first order ODE on $U \subset \mathbb{R}^k$:

$$x'(t) = F_1(\psi(x(t))) = F_1(x(t), f(x(t))), \quad x(0) = p_1$$

where $F = (F_1, F_2)$ and $p = (p_1, p_2)$ are written as a pair of $k$-vectors and $(n-k)$-vectors, i.e. $F_1$ and $p_1$ are the first $k$ coordinates of $F$ and $p$ respectively.

Since $F$ is smooth on $\mathbb{R}^n$, so does $F_1$. Because $M$ is compact, $F_1$ and its derivatives are uniformly bounded on $M$. This implies that $F_1$ is Lipschitz and thus we can apply the existence and uniqueness theorem for first order ODE to conclude that the system (*) has a unique solution around a small neighborhood of $t = 0$. Suppose $x(t)$ is the solution for $t \in (-\epsilon, \epsilon)$. We claim that this solution lifts to a solution $y(t) = (x(t), f(x(t))) = \psi(x(t))$ for the original system of ODE. The initial condition is obviously satisfied since $y(0) = (x(0), f(x(0))) = (p_1, f(p_1)) = p$. To see that $y'(t) = F(y(t))$, it requires a bit more work.

By chain rule, we have $y'(t) = (x'(t), Df_{x(t)}(x'(t))) = (F_1(x(t), f(x(t))), Df_{x(t)}(x'(t)))$. Therefore, it suffices to prove that $Df_{x(t)}(x'(t)) = F_2(x(t), f(x(t)))$ for all $t \in (-\epsilon, \epsilon)$. Since $F(y)$ is tangent to $M$ at $y$ for all $y \in M$, this means that for any fixed $t \in (-\epsilon, \epsilon)$, there is a curve $\gamma(t) : I \to U$ such that $\psi(\gamma(0)) = y(t)$ and $\frac{d}{ds}\psi(\gamma(s))|_{s=0} = F(y(t))$. Equivalently, $(\gamma(0), f(\gamma(0))) = y(t) = (x(t), f(x(t)))$, thus $\gamma(0) = x(t)$, and $(\gamma'(0), Df_{\gamma(0)}(\gamma'(0))) = F(x(t), f(x(t)))$, thus $Df_{x(t)}(F_1(x(t), f(x(t)))) = Df_{x(t)}(x'(t)) = F_2(x(t), f(x(t)))$. Therefore, we have shown that $y(t)$ actually solves the original system on $t \in (-\epsilon, \epsilon)$.

The last thing we have to show is that this solution actually extends to all $t \in \mathbb{R}$. Suppose $t_0 = \sup\{t > 0 : y(t) \text{ can be extended to the interval } (-\epsilon, t)\}$.

We claim that $t_0 = \infty$. Suppose not. Then $t_0$ is some finite positive real number. Since $M$ is compact, $F$ is bounded on $M$ and hence $y(t)$ is bounded and has bounded derivatives. Therefore $y(t)$ is Lipschitz and uniformly continuous on $(-\epsilon, t_0)$, so it can be extended continuously to $[-\epsilon, t_0]$. By uniqueness and existence theorem applied to the initial point at $y(t_0)$, we know that $y(t)$ actually extends further, say to $t_0 + \delta$, which contradicts the maximality of $t_0$. Therefore, $t_0 = \infty$. A similar argument shows that $y(t)$ can be extended to any negative $t$. So we are done.

**Problem 4. (Taylor Ex.I.3.1)** Suppose that $F : U \to \mathbb{R}^n$ is a $C^2$-map, $U$ is open in $\mathbb{R}^n$, $p \in U$, and $DF(p)$ is invertible. With $q = F(p)$, define a map $N$ on a neighborhood of $p$ by

$$N(x) = x + DF(x)^{-1}(q - F(x)).$$

Show that there exists $\epsilon > 0$ and $C < \infty$ such that, for $0 \leq r < \epsilon$,

$$\|x - p\| \leq r \Rightarrow \|N(x) - p\| \leq Cr^2.$$
Conclude that if \( \| x_1 - p \| \leq r \), with \( r < \min(\epsilon, 1/2C) \), then \( x_{j+1} = N(x_j) \) defines a sequence converging very rapidly to \( p \). This is the basis of Newton’s method, for solving \( F(p) = q \) for \( p \). (Hint: Write \( x = p + y \), \( F(x) = F(p) + DF(x)y + R \), with \( R \) given as in (1.27), with \( k = 2 \). Then \( N(x) = p + \overline{y} \), \( \overline{y} = -DF(x)^{-1}R \).)

**Solution.** Write \( x = p + y \), doing Taylor series expansion up to the first order term:

\[
F(x) = F(p) + DF(x)y + R(y)
\]

where

\[
R(y) = \sum_{|\alpha| = 2} \frac{2}{\alpha!} \left( \int_0^1 (1 - s) F^{(\alpha)}(sy) \, ds \right) y^\alpha.
\]

Hence, \( N(x) = x + DF(x)^{-1}(F(p) - F(x)) = p - DF(x)^{-1}(R(y)) \). Since \( F \) is \( C^2 \) and \( DF(p) \neq 0 \), we can choose a small \( \epsilon > 0 \) such that \( DF(x) \neq 0 \) and \( \|DF(x)^{-1}\| \leq C \) for all \( \|x - p\| < \epsilon \) for some \( C > 0 \). Moreover it is easy to see from the expression of \( R(y) \) that \( \|R(y)\| \leq Cy^2 \) for some \( C > 0 \). Thus, for any \( 0 \leq r < \epsilon \) and \( \|x - p\| \leq r \),

\[
\|N(x) - p\| \leq \|DF(x)^{-1}\||R(y)|| \leq Cr^2.
\]

If \( \|x_1 - p\| \leq r \), with \( r < \min(\epsilon, 1/2C) \), then

\[
\|x_2 - p\| = \|N(x_1) - p\| \leq Cr^2 \leq \frac{1}{2} r.
\]

An induction argument show that \( \|x_n - p\| \leq \frac{1}{2^n} r \) for all \( n \), which converges to 0 exponentially fast. Therefore, \( x_n \) converges to \( p \) rapidly provided that our initial guess \( x_1 \) is close enough to \( p \).

**Problem 5.** (Taylor Ex.1.3.4) Let \( \mathcal{O} \subset \mathbb{R}^n \) be open. We say that a function \( f \in C^\infty(\mathcal{O}) \) is real analytic provided that for each \( x_0 \in \mathcal{O} \), we have a convergent power-series expansion

\[
f(x) = \sum_{\alpha \geq 0} \frac{1}{\alpha!} f^{(\alpha)}(x_0) (x - x_0)^\alpha,
\]

valid in a neighborhood of \( x_0 \). Show that we can let \( x \) be complex in (3.14), and obtain an extension \( f \) to a neighborhood of \( \mathcal{O} \) in \( \mathbb{C}^n \). Show that the extended function is holomorphic, that is, satisfies the Cauchy-Riemann equations. **Remark.** It can be shown that, conversely, any holomorphic function has a power-series expansion. See (2.30) of Chapter 3 for one such proof. For the next exercise, assume this to be known.

**Solution.** Since a convergent power series actually converges absolutely and uniformly on some small neighborhood around the point of expansion, therefore, allowing \( x \) to be complex numbers close enough to \( x_0 \), the series actually converges absolutely. This defines an extension of \( f \).
Next observe that uniform convergence implies that we can put \( \frac{\partial}{\partial x^i} \) \((i = 1, \ldots, n)\) into the summation sign and hence
\[
\frac{\partial}{\partial x^i} f(x) = \sum_{\alpha \geq 0} \frac{1}{\alpha!} f^{(\alpha)}(x_0) \frac{\partial}{\partial x^i}(x - x_0)^\alpha.
\]
Because one can check that polynomials are holomorphic, we have \( \frac{\partial}{\partial x^i}(x - x_0)^\alpha = 0 \) for all \( \alpha \). Hence \( f \) is analytic in a small neighborhood of \( x_0 \).

Remark: Recall that the Cauchy Riemann equations for a function \( f(z) \) to be analytic can be written as \( \frac{\partial}{\partial z} f(z) = 0 \) where \( \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \) with \( z = x + iy \).

**Problem 6.** *(Taylor Ex.I.4.3)* Let \( X \) be an \( n \times n \) matrix. Show that \[
\det e^X = e^{Tr X}.
\]
(Hint: Use a normal form.)

**Solution.** Choose a basis such that \( X \) is upper triangular, i.e. choose an invertible matrix \( Q \) so that \( X = QYQ^{-1} \), where \( Y \) is upper triangular. An elementary calculation shows that \( e^X = Qe^YQ^{-1} \). Observe that
\[
Y^k = \begin{pmatrix} y_{11}^k & * & \cdots & * \\
* & y_{22}^k & \cdots & * \\
& \ddots & \ddots & \ddots \\
& & \ddots & y_{nn}^k
\end{pmatrix}.
\]
Therefore,
\[
e^Y = \begin{pmatrix} e^{y_{11}} & * & \cdots & * \\
* & e^{y_{22}} & \cdots & * \\
& \ddots & \ddots & \ddots \\
& & \ddots & e^{y_{nn}}
\end{pmatrix}.
\]
Taking determinant gives \( \det e^X = \det e^Y = e^{Tr Y} = e^{Tr X} \).

**Problem 7.** *(Taylor Ex.I.4.4)* Let \( M_n \) be the space of complex \( n \times n \) matrices. If \( A \in M_n \), and \( \det A = 1 \) we say that \( A \in SL(n, \mathbb{C}) \). If \( X \in M_n \) and \( Tr X = 0 \), we say that \( X \in \text{sl}(n, \mathbb{C}) \). Let \( X \in \text{sl}(2, \mathbb{C}) \). Suppose \( X \) has eigenvalues \( \{ \lambda, -\lambda \} \), \( \lambda \neq 0 \). Such an \( X \) can be diagonalized, so we know that there exist matrices \( Z_j \in M_2 \) such that
\[
e^{tX} = Z_1 e^{t\lambda} + Z_2 e^{-t\lambda}.
\]
Evaluating both sides at \( t = 0 \), and the \( t \)-derivative at \( t = 0 \), show that \( Z_1 + Z_2 = I \), \( \lambda Z_1 - \lambda Z_2 = X \), and solve for \( Z_1, Z_2 \). Deduce that
\[
e^{tX} = (\cosh t\lambda)I + \lambda^{-1}(\sinh t\lambda)X.
\]
Solution. We know that $e^{tX} = Z_1 e^{t\lambda} + Z_2 e^{-t\lambda}$. Differentiate this expression gives

$Xe^{tX} = \lambda Z_1 e^{t\lambda} - \lambda Z_2 e^{-t\lambda}$.

Evaluate at $t = 0$ gives a system of linear equations:

\begin{align*}
Z_1 + Z_2 &= I \\
\lambda Z_1 - \lambda Z_2 &= X
\end{align*}

Solving it, we have

\begin{align*}
Z_1 &= \frac{1}{2}(I + \lambda^{-1}X) \\
Z_2 &= \frac{1}{2}(I - \lambda^{-1}X)
\end{align*}

Put it back into the equation $e^{tX} = Z_1 e^{t\lambda} + Z_2 e^{-t\lambda}$, we have

$e^{tX} = (\cosh t\lambda)I + \lambda^{-1}(\sinh t\lambda)X$. 