MATH 174A: FINAL EXAM
THURSDAY, MARCH 22, 2007

There are three problems. Do all of them.

Problem 1. Consider the vector field $X(x, y) = (-y, x) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ in $\mathbb{R}^2$.

(1) Find all integral curves of X.
(2) Show that the equation $Xu = 0$, $u \in C^1(\mathbb{R}^2)$, with initial condition $u(0, y) = f(y)$, $y > 0$, f given C^1 function vanishing near 0, has a unique solution.
(3) Can $Xu = 0$, $u \in C^1(\mathbb{R}^2)$, with initial condition $u(0, y) = f(y)$, $y \in \mathbb{R}$, f given C^1 function vanishing near 0, be solved in general?

Solution:

(1) To find the integral curves of X, we need to solve the following system:
$$\begin{cases} x' = -y \\ y' = x \end{cases}.$$
Hence,
$$(x^2 + y^2)' = 2xx' + 2yy' = 2x(-y) + 2y(x) = 0.$$
Therefore, the integral curves are concentric circles centered at 0, i.e. $x^2 + y^2 = r^2$, $r \geq 0$.
(2) If $u \in C^1(\mathbb{R}^2)$ solves the PDE $Xu = 0$. Then u must be constant on each integral curves. Since each integral curve (except the origin) intersects with the ray $y > 0$ at exactly one point, so the "initial" condition uniquely determines the solution u (if exists) on $\mathbb{R}^2 \setminus \{0\}$. Let $u(x, y) = f(\sqrt{x^2 + y^2})$. Note that $f \in C^1$ and f vanishes on a neighborhood of 0. So u is in $C^1(\mathbb{R}^2)$ and solves the PDE $Xu = 0$ since it is constant on each integral curve.
(3) No, by the argument above, u has to be constant on each integral curve. So, if $f(y) \neq f(-y)$ for some $y > 0$, then there is no solution to the PDE with this initial condition.

Problem 2. Consider the heat equation $u_t = ku_{\theta\theta}$ is a thin ring, where $u(t, \theta)$ is the temperature at time t, place θ, and suppose that the initial temperature of the ring is $u(\theta, 0) = \phi(\theta)$, $\theta \in \mathbb{S}^1$, ϕ is a given function in $C^1(\mathbb{S}^1)$.

(1) Using the separation of variables, show that the solution of the heat equation is of the form
$$\sum_{n \in \mathbb{Z}} A_n e^{-n^2 kt} e^{in\theta},$$
and find A_n in terms of ϕ.
(2) The total heat energy in the ring at time t is
$$Q(t) = \int_0^{2\pi} u(\theta, t)\, d\theta.$$

Using the PDE, show that $Q(t)$ is a constant (independent of t). (Hint: consider dQ/dt.)

(3) Find Q in terms of the A_n. What does the conservation of heat energy, as in (2), correspond to in the series solution (1)?

(4) Find $\lim_{t\to+\infty} u(\theta,t)$ in terms of ϕ.

Solution:

(1) Let $u(\theta,t)=\Theta(\theta)T(t)$. Plug into the heat equation $u_t = ku_{\theta\theta}$, we get

$$\frac{\Theta''}{\Theta}=\frac{T'}{kT}=-\lambda.$$

Solving the ODE

$$\begin{cases}\Theta''+\lambda\Theta=0\\ \Theta(0)=\Theta(2\pi),\end{cases}$$

we get $\lambda=n^2$ $(n\in\mathbb{Z})$ and $\Theta_n(\theta)=e^{in\theta}$. On the other hand, solving the ODE involving T gives us $T_n(t)=e^{-n^2kt}$. Therefore, the general solution is

$$u(\theta,t)=\sum_{n\in\mathbb{Z}} A_n e^{-n^2kt}e^{in\theta}.$$

To determine the constants A_n, we look at the initial condition $u(\theta,0)=\phi(\theta)$. We get

$$A_n=\frac{1}{2\pi}\int_0^{2\pi}\phi(\theta)e^{-in\theta}\,d\theta,$$

which is the Fourier coefficients of the C^1 function ϕ.

(2)

$$\frac{dQ}{dt}=\frac{d}{dt}\int_0^{2\pi}u(\theta,t)d\theta=\int_0^{2\pi}u_t d\theta=\int_0^{2\pi}ku_{\theta\theta}d\theta$$
$$=k(u_\theta(2\pi)-u_\theta(0))=0,$$

since u is 2π-periodic in θ. Therefore, $Q(t)$ is constant.

(3) Since ϕ is C^1, its Fourier series is absolutely convergent and it converges uniformly to ϕ. So, for each fixed $t>0$, the series solution in (1) is absolutely and uniformly convergent. So

$$Q(t)=\int_0^{2\pi}u d\theta=\sum_{n\in\mathbb{Z}}A_n e^{-n^2kt}\int_0^{2\pi}e^{in\theta}d\theta.$$

Since $\int_0^{2\pi}e^{in\theta}d\theta=0$ except for $n=0$, so we have

$$Q(t)=2\pi A_0.$$

The conservation of energy in (2) corresponds to the fact that the series solution given in (1) is 2π- periodic in θ (loosely speaking, the system is closed).

(4) Again, by uniform convergence of the series and the fact that $\lim_{t\to+\infty}e^{-n^2kt}=0$ for all $n\neq 0$, we have

$$\lim_{t\to+\infty}u(\theta,t)=\sum_{n\in\mathbb{Z}}A_n\left(\lim_{t\to+\infty}e^{-n^2kt}\right)e^{in\theta}=A_0.$$

Problem 3. Recall that $s(\mathbb{Z})$ is the set of rapidly decreasing bi-infinite sequences, i.e. sequences $a = \{a_n\}$ such that $\|a\|_k = \sup(1+|n|)^k |a_n| < \infty$ for all $k \geq 0$ integer, and we make $s(\mathbb{Z})$ into a complete metric space using these norms.

Recall also that a linear map $u : s(\mathbb{Z}) \to \mathbb{C}$ is continuous if and only if there exists $C > 0$ and k such that $|u(a)| \leq C\|a\|_k$ for all $a \in s(\mathbb{Z})$, and an analogous statement holds for all metric spaces where the metric is constructed from a sequence of norms, e.g. $C^\infty(\mathbb{S}^1)$.

(1) Show that if $a = \{a_n\} \in s(\mathbb{Z})$, and $a^{(k)} \in s(\mathbb{Z})$ is the sequence with $a_n^{(k)} = 0$ if $|n| > k$, $a_n^{(k)} = a_n$ if $|n| \leq k$, then $\lim_{k\to\infty} a^{(k)} = a$ in $s(\mathbb{Z})$.

(2) Show that if $b = \{b_n\}$ is a polynomially bounded sequence, i.e. there exists N and C such that $|b_n| \leq C(1+|n|)^N$ for all $n \in \mathbb{Z}$, then b defines a continuous linear functional on $s(\mathbb{Z})$ by $b(\{a_n\}) = \sum_{n\in\mathbb{Z}} b_n a_n$.

(3) Show that if $b(a) = 0$ for all $a \in s(\mathbb{Z})$ then $b = 0$.

(4) Conversely, suppose that $u : s(\mathbb{Z}) \to \mathbb{C}$ is a continuous linear map. Let $\epsilon^{(k)} \in s(\mathbb{Z})$ be the sequence whose kth entry is 1, and all others are 0: $\epsilon_n^{(k)} = 1$ if $k = n$, $\epsilon_n^{(k)} = 0$ if $k \neq n$. Let $b_n = u(\epsilon^{(n)})$. Show that $\{b_n\}$ is polynomially bounded, and $u(\{a_n\}) = \sum b_n a_n$ for all $a = \{a_n\} \in s(\mathbb{Z})$. (Hint: For the last claim, show it first if a has only finitely many non-zero entries.)

   Deduce that the set of continuous linear maps on $s(\mathbb{Z})$ can be identified with the set $s'(\mathbb{Z})$ of polynomially bounded sequences.

(5) Let $\mathcal{D}'(\mathbb{S}^1)$ be the set of continuous linear maps (distributions) $C^\infty(\mathbb{S}^1) \to \mathbb{C}$. Show that the Fourier series map $\mathcal{F} : C^\infty(\mathbb{S}^1) \to s(\mathbb{Z})$ extends to a map $\mathcal{F} : \mathcal{D}'(\mathbb{S}^1) \to s'(\mathbb{Z})$ by letting $(\mathcal{F}u)_n = (2\pi)^{-1} u(e_n)$, $e_n(x) = e^{-inx}$, $n \in \mathbb{Z}$. (That is, you must show that $\mathcal{F}$ indeed gives a map as stated, and if $u = \iota_f$, $f \in C^\infty(\mathbb{S}^1)$, i.e. $u(\phi) = \int_{\mathbb{S}^1} f(x)\phi(x)\,dx$, $\phi \in C^\infty(\mathbb{S}^1)$, then the above definition agrees with the standard one for the Fourier coefficients $\mathcal{F}f$ of f.)

(6) Find $\mathcal{F}\delta_\omega$, $\omega \in \mathbb{S}^1$. (Recall: $\delta_\omega(\phi) = \phi(\omega)$.)

(7) Recall that for $u \in \mathcal{D}'(\mathbb{S}^1)$, $\frac{du}{dx} \in \mathcal{D}'(\mathbb{S}^1)$ is given by $\frac{du}{dx}(\phi) = -u(\frac{d\phi}{dx})$, $\phi \in C^\infty(\mathbb{S}^1)$. Show that $(\mathcal{F}\frac{du}{dx})_n = in(\mathcal{F}u)_n$, $u \in \mathcal{D}'(\mathbb{S}^1)$.

(8) Show that $\mathcal{F} : \mathcal{D}'(\mathbb{S}^1) \to s'(\mathbb{Z})$ is injective, i.e. that if $\mathcal{F}u = 0$, i.e. $u(e_n) = 0$ for all n, then $u = 0$, by using that the Fourier series of a C^∞ function converges to the function in C^∞ (i.e. uniformly with all derivatives).

(9) Use this to show that if $u \in \mathcal{D}'(\mathbb{S}^1)$ and $\frac{du}{dx} = 0$, then u is the distribution associated to a constant function.

Solution:

(1) We need to show that $a^{(k)}$ converges to a in $\|.\|_m$ for all m. Since $a \in s(\mathbb{Z})$, for each $m \in \mathbb{N}$, there is a constant C_m such that

$$|a_n| \leq \frac{C_m}{(1+|n|)^m}$$

   for all $n \in \mathbb{Z}$. So for each $m, k \in \mathbb{N}$, we have

$$\|a^{(k)} - a\|_m = \sup_{|n|>k} (1+|n|)^m |a_n| \leq \sup_{|n|>k} (1+|n|)^m \frac{C_{m+1}}{(1+|n|)^{m+1}} \leq \frac{C_{m+1}}{1+k}.$$

   Take $k \in \infty$ yields the desired result $\lim_{k\to\infty} a^{(k)} = a$.

(2) We can actually take $C_m = \|a\|_m$ in part (1). So

$$|b(a)| \leq \sum_n |b_n||a_n| \leq C \sum_n (1+|n|)^N \frac{\|a\|_{N+2}}{(1+|n|)^{N+2}}$$

$$\leq C\|a\|_{N+2} \sum_n \frac{1}{(1+|n|)^2}.$$

Since the series $\sum 1/(1+|n|)^2$ converges, we have $|b(a)| \leq C'\|a\|_{N+2}$ for some constant $C' > 0$. Hence, b is a continuous functional on $s(\mathbb{Z})$.

(3) For each $i \in \mathbb{Z}$, let a be sequence with 1 in the i-th term and 0 elsewhere. Clearly $a \in s(\mathbb{Z})$ and $b(a) = b_i$. Since $b(a) = 0$ for all $a \in s(\mathbb{Z})$ by assumption, we have $b_i = 0$. Therefore, $b = 0$.

(4) By definition, u is a continuous linear functional on $s(\mathbb{Z})$ means that there exist $C, N > 0$ such that $|u(a)| \leq C\|a\|_N$. Hence

$$|b_n| = |u(\epsilon^{(n)})| \leq C\|\epsilon^{(n)}\|_N = C(1+|n|)^N,$$

i.e. b is polynomially bounded. To prove the formula $u(a_n) = \sum b_n a_n$, first we check this for finite sequence a_n, i.e. there is an $N > 0$ such that $a_n = 0$ for $|n| > N$.Then

$$u(a_n) = u\left(\sum_{|n|\leq N} a_n \epsilon^{(n)}\right) = \sum_{|n|\leq N} a_n u(\epsilon^{(n)}) = \sum_{|n|\leq N} a_n b_n = \sum_{n\in\mathbb{N}} a_n b_n.$$

For an arbitrary sequence $a \in s(\mathbb{Z})$, by (1), we know that the truncated sequence $a^{(k)}$ converges to a in $s(\mathbb{Z})$. By continuity of u,

$$u(a) = \lim_{k\to\infty} u(a^{(k)}) = \lim_{k\to\infty} \sum_{|n|\leq k} a_n b_n = \sum_{n\in\mathbb{Z}} a_n b_n.$$

Define the correspondence by sending each polynomially bounded sequence b to the continuous linear functional u_b, where $u_b(a) = \sum a_n b_n$. What we have shown above tells us that this correspondence is surjective and the result in part (3) implies injectivity. Therefore, this is a bijective correspondence.

(5) We first show that the map $\mathcal{F}$ so defined really defines a map from $\mathcal{D}'(\mathbb{S}^1)$ to $s'(\mathbb{Z})$. Let u be a continuous linear functional on $C^\infty(\mathbb{S}^1)$. By continuity, we can find $C, k > 0$ such that $|u(f)| \leq C\|f\|_{C^k}$ for all $f \in C^\infty(\mathbb{S}^1)$. Therefore

$$|\mathcal{F}u_n| = (2\pi)^{-1}|u(e_n)| \leq C\|e_n\|_{C^k} = C(1+|n|+|n|^2+\cdots+|n|^k)$$

$$\leq C(1+|n|)^k.$$

Therefore, $\{(\mathcal{F}u)_n\} \in s'(\mathbb{Z})$. Then we show that this actually extends the original Fourier map. Let $f \in C^\infty(\mathbb{S}^1)$, and ι_f be its corresponding distribution as defined in the hint. Then

$$(\mathcal{F}\iota_f)_n = \frac{1}{2\pi}\iota_f(e_n) = \frac{1}{2\pi}\int_{\mathbb{S}^1} f(x)e_n(x)dx = \mathcal{F}f_n,$$

the usual Fourier coefficients of f.

(6)

$$(\mathcal{F}\delta_\omega)_n = (2\pi)^{-1}\delta_\omega(e_n) = (2\pi)^{-1}e^{-in\omega}.$$

(7)

$$(\mathcal{F}\frac{du}{dx})_n = (2\pi)^{-1}\frac{du}{dx}(e_n) = -(2\pi)^{-1}u(\frac{d}{dx}e_n) = in(2\pi)^{-1}u(e_n) = in(\mathcal{F}u)_n.$$

(8) Let $f \in C^\infty(\mathbb{S}^1)$ and $\sum a_n e_n$ be its Fourier series which converges to f in C^∞. Therefore, by linearity and continuity of u, $u(f) = \sum a_n u(e_n) = 0$ since $u(e_n) = 0$ for all n. Therefore, $\mathcal{F}$ is injective.

(9) If $\frac{du}{dx} = 0$. then by (7), for all n,

$$0 = (\mathcal{F}\frac{du}{dx})_n = in(\mathcal{F}u)_n.$$

This means that $(\mathcal{F}u)_n = 0$ for all n except possibly at $n = 0$. Suppose $(\mathcal{F}u)_0 = a_0$. Let $f \equiv a_0$ be the constant function. Then $(\mathcal{F}\iota_f)_n = 0$ for all $n \neq 0$ and $(\mathcal{F}\iota_f)_0 = a_0$. Therefore, $\mathcal{F}u = \mathcal{F}\iota_f$. Since $\mathcal{F}$ is injective by (8), we have $u = \iota_f$, which is a distribution associated to a constant function f.