MATH 174A: MIDTERM
DUE AT 4PM ON MONDAY, FEBRUARY 12, 2007

Problem 1. *The purpose of this problem is to see how our results so far help us solve first order PDEs. Suppose O is an open subset of $\mathbb{R}^n$, and consider a scalar semilinear PDE on O, i.e. suppose that X is a C^1 vector field on O (so $X : O \to \mathbb{R}^n$), $c : O \times \mathbb{R} \to \mathbb{R}$ is C^1, and consider the PDE*

$$X(x)u(x) = c(x, u(x)),\ u \in C^1(O; \mathbb{R}).$$

Suppose also that M is a hypersurface such that for $p \in M \cap O$, $X(p)$ is not in the tangent space of M, and we want $u|_M = w$, where w is a given C^1 function.

(1) *Show that if γ is an integral curve of X and $v = u \circ \gamma$ satisfies $v'(t) = c(\gamma(t), v(t))$, then the PDE is satisfied at $x = \gamma(t)$. Show also conversely, if the PDE holds, then the ODE $v'(t) = c(\gamma(t), v(t))$ must be satisfied for all integral curves.*
(2) *Show that the PDE has a unique C^1 solution u near M. (Hint: 'straighten out' the vector field X.)*
(3) *Show that if X and c are C^k, then u is also C^k.*
(4) *Solve the following PDE explicitly:*

$$u_x + xu_y = u^2,\ u(0, y) = \sin y.$$

Show that while the solution exists for $|x|$ small, it does not extend to all of $\mathbb{R}^2$. (Hint: You need not straighten out the vector field. Find the integral curves directly.)

Solution. (1) Let $v = u \circ \gamma$. By the chain rule,

$$v'(t) = (Du)(\gamma(t))\gamma'(t) = (Du)(\gamma(t))X(\gamma(t)) = (Xu)(\gamma(t)).$$

So the PDE holds at $x = \gamma(t)$ if and only if $v'(t) = c(\gamma(t), u(\gamma(t)) = c(\gamma(t), v(t))$.

(2) There is a C^1 diffeomorphism Φ 'straightening out the flow' near $p \in M$, so $\Phi : O \to U$, $U \subset \mathbb{R}^n$, O a neighborhood of p in $\mathbb{R}^n$, $\Phi_* X = \frac{\partial}{\partial z_n}$, where we write coordinates on U as $(z_1, \ldots, z_n)$, and M is mapped to $z_n = 0$. Letting $\tilde{u} = u \circ \Phi^{-1}$,

$$\frac{\partial \tilde{u}}{\partial z_n} = (\Phi_* X)\tilde{u} = Xu,$$

so the original PDE holds if and only if $\tilde{u}$ satisfies

$$\frac{\partial \tilde{u}}{\partial z_n} = \tilde{c}(z, \tilde{u}(z)),$$

where $\tilde{c}(z, y) = c(\Phi^{-1}(z), y)$. But this is simply an ODE for each fixed value of $(z_1, \ldots, z_{n-1})$, with initial condition

$$\tilde{u}(z_1, \ldots, z_{n-1}, 0) = w(\Phi^{-1}((z_1, \ldots, z_{n-1}, 0))),$$

so by the ODE existence theorem, it has a unique solution, and this solution depends in a C^1 manner on the initial conditions as well, hence on z. Thus, $u = \tilde{u} \circ \Phi$ is the unique C^1 solution of the original problem.

(3) If X is C^k, the diffeomorphism Φ is also C^k. The regularity theorem for ODEs states that the solutions are C^k functions of 'time' (in this case, z_n), as well as the initial conditions. As Φ^{-1} is also C^k by the inverse function theorem, the initial conditions are C^k functions of $(z_1, \ldots, z_{n-1})$, so $\tilde{u}$ is C^k, hence u itself is C^k.

(4) The integral curves of $X(x, y) = (1, x)$ satisfy

$$\frac{dx}{dt} = 1, \ \frac{dy}{dt} = x.$$

Solving them so at $t = 0$, $x(0) = 0$, $y(0) = y_0$ (so we are calculating the integral curve through $(0, y_0)$) gives

$$x = t \Rightarrow y = \frac{t^2}{2} + y_0.$$

Then along $\gamma(t) = (x(t), y(t))$, we have $v = u \circ \gamma$ satisfies $v'(t) = v^2$, so if $v(0) \neq 0$,

$$\frac{1}{v(0)} - \frac{1}{v} = t,$$

while if $v(0) = 0$, then $v(t) = 0$ for all t. Thus,

$$v(t) = \frac{v(0)}{1 - v(0)t},$$

valid in either case. For us, $v(0) = w(0, y_0) = \sin y_0$, and $y_0 = y - \frac{t^2}{2} = y - \frac{x^2}{2}$, so

$$u(x, y) = \frac{\sin(y - x^2/2)}{1 - x\sin(y - x^2/2)}.$$

Note that the solution is certainly well-defined for $|x|$ small, indeed if $|x| < 1$, but if $x = 1$, and $\sin(y - x^2/2) = 1$, e.g. $y = \frac{1}{2} + \frac{\pi}{2}$, then the numerator is non-zero, the denominator vanishes, hence $|u(x, y)| \to \infty$ as (x, y) approaches $(1, \frac{1}{2} + \frac{\pi}{2})$.

Problem 2. *Suppose M is a compact k-dimensional surface in $\mathbb{R}^n$, and h is a C^∞ function on $\mathbb{R}^n$. Then*

$$\nabla h = \left(\frac{\partial h}{\partial x_1}, \ldots, \frac{\partial h}{\partial x_n}\right)$$

is a C^∞ vector field on $\mathbb{R}^n$. For $p \in M$, let $X(p) \in T_pM$ be the orthogonal projection of $\nabla h(p)$ to T_pM, i.e. $\nabla h(p) - X(p) \in (T_pM)^\perp$. Let $\mathcal{F}_X^t$ denote the flow of X at time t.

(1) *Show that for $p \in M$ and $v \in T_pM$, $X(p) \cdot v = Dh(p)v$, where $\cdot$ on the left hand side is the standard inner product on $\mathbb{R}^n$.*

(2) *Show that for all $p \in M$, $h(\mathcal{F}_X^t(p))$ is an increasing function of t.*

(3) *We say that a point $q \in M$ is a critical point of $h|_M$ if $Dh(q)v = 0$ for $v \in T_qM$. Show that the critical points of $h|_M$ are exactly the critical points of X (i.e. the points q where $X(q) = 0$).*

(4) *Suppose that there are finitely many critical points of* $h|_M$*. Show that for all* $p \in M$*,* $\lim_{t\to+\infty} \mathcal{F}_X^t(p)$ *exists (and similarly for* $t \to -\infty$*) and is a critical point of* h*. (Hint: consider* $\gamma_n(t) = \mathcal{F}_X^{t+T_n}(p)$*. For each* n*,* γ_n *is an integral curve of* X*. What can you say about these if* $\mathcal{F}_X^{T_n}(p)$ *converges?)*

(5) *If* $M = \mathbb{S}^{n-1}$ *is the unit sphere given by* $M = \{x \in \mathbb{R}^n : \|x\| = 1\}$ *and* $h(x) = x_n$*, find* $\lim_{t\to+\infty} \mathcal{F}_X^t(p)$ *and* $\lim_{t\to-\infty} \mathcal{F}_X^t(p)$ *for all* $p \in M$*.*

The flow of $-X$ *is called the gradient flow of* h*; the flow of* X *is thus the time-reversed gradient flow.*

Note: if you prefer to have X *defined near* M*, rather than just on* M*, near* $p_0 \in M$ *you can take* $\Phi : \mathbb{R}^n \to \mathbb{R}^{n-k}$*, as in the definition of a surface, such that* $D\Phi(p_0)$ *is surjective and* M *is the set of points with* $\Phi = 0$*. Then the surfaces* $M_c = \{x : \Phi(x) = c\}$*,* $c \in \mathbb{R}^{n-k}$*, can be used (locally) to define* X*:* $X(q)$ *is the orthogonal projection of* ∇h *to* $T_q M_c$*, with* $c = \Phi(q)$*.*

Solution. First, we remark that X is indeed a C^∞ vector field. To see this, recall from the homework that if M is locally defined by $\Phi = 0$ on an open set O where $\Phi = (\Phi_1, \dots, \Phi_{n-k})$, then T_pM is the orthocomplement of the linear span S of $(\nabla\Phi_1(p), \dots, \nabla\Phi_{n-k}(p))$. The surjectivity of $D\Phi(p)$ means that the $\nabla\Phi_j(p)$, $j = 1, \dots, n-k$, are linearly independent. The Gram-Schmidt procedure gives an orthonormal set $e_1(p), \dots, e_{n-k}(p)$ with span equal to S. As the Gram-Schmidt procedure is given by explicit formulae, it is straightforward to see that (locally) the e_j are smooth maps $e_j : O \to \mathbb{R}^n$. The orthogonal projection of a vector v onto S is

$$\sum_{j=1}^{n-k} (e_j(p) \cdot v) e_j(p),$$

so

$$X(p) = \nabla h(p) - \sum_{j=1}^{n-k} (e_j(p) \cdot \nabla h(p)) e_j(p),$$

As the e_j as well as h are C^∞, we deduce that X is C^∞ as well.

(1) As $\nabla h - X(p) \in (T_pM)^\perp$, for $v \in T_pM$, $(\nabla h - X(p)) \cdot v = 0$, i.e. $\nabla h \cdot v = X(p) \cdot v$. But $\nabla h \cdot v = \sum_{j=1}^n \frac{\partial h}{\partial x_j} v_j = Dh(p)v$, so $Dh(p)v = X(p) \cdot v$.

(2) Let $\gamma(t) = \mathcal{F}_X^t(p)$. Then

$$(h \circ \gamma)'(t) = Dh(\gamma(t))\gamma'(t) = Dh(\gamma(t))X(\gamma(t)) = X(\gamma(t)) \cdot X(\gamma(t)) \geq 0,$$

where we use part (1) with $v = X(\gamma(t))$ (which is tangent to M at $\gamma(t)$). Thus, $h \circ \gamma$ is an increasing function of t.

(3) By part (1), $Dh(q)v = 0$ for all $v \in T_qM$ if and only if $X(q) \cdot v = 0$ for all $v \in T_qM$. But $X(q) \in T_qM$, so $X(q) \cdot w = 0$ if $w \in (T_qM)^\perp$. Thus, $X(q) \cdot v = 0$ for all $v \in T_qM$ if and only if $X(q) \cdot v = 0$ for all $v \in \mathbb{R}^n$. As one can take $v = X(q)$ in particular, this holds if and only if $X(q) = 0$.

(4) Consider the sequence $t_n = n$. As M is compact, the sequence $\{\gamma(t_n)\}_{n=1}^\infty$ has a convergent subsequence, say $\{\gamma(T_n)\}$; let $q = \lim_{n\to\infty} \gamma(T_n)$. Thus, T_n is increasing, and $T_n \to +\infty$ as $n \to \infty$. As solutions of ODEs depend continuously on the initial condition, the solution γ_n of the ODE $\gamma_n'(t) = X(\gamma_n(t))$, $\gamma_n(0) = \gamma(T_n)$ converges uniformly (over compact time intervals) to the solution $\tilde\gamma$ of the ODE

$$\tilde\gamma'(t) = X(\tilde\gamma(t)),\ \tilde\gamma(0) = \lim_{n\to\infty} \gamma(T_n) = q.$$

Also, as h is continuous, $\lim_{n\to\infty} h(\gamma(T_n)) = q$. Now note that $h \circ \gamma$ is increasing, hence $h(\gamma(T_n))$ is an increasing function of n, we have for $t \in [T_n, T_{n+1}]$ that $h(\gamma(T_n)) \leq h(\gamma(t)) \leq h(\gamma(T_{n+1}))$, so if N is such that $n \geq N$ implies $|h(\gamma(T_n)) - h(q)| < \epsilon$ then for $t \geq T_N$, $|h(\gamma(t)) - h(q))| < \epsilon$. In other words, $\lim_{t\to+\infty} h(\gamma(t)) = h(q)$. As $\tilde{\gamma}(t) = \lim_{n\to\infty} \gamma(T_n + t)$, we deduce that $h(\tilde{\gamma}(t)) = h(q)$ for all t. Thus, $\frac{d}{dt}(h \circ \tilde{\gamma}) = 0$. But

$$\frac{d}{dt} h(\tilde{\gamma}(t)) = \|X(\gamma(t))\|^2,$$

so $X(\tilde{\gamma}(t)) = 0$ for all t. Hence $\tilde{\gamma}'(t) = 0$ for all t, so $\tilde{\gamma}(t) = \tilde{\gamma}(0) = q$ for all t, i.e. $\tilde{\gamma}$ is the constant curve at q, and $X(q) = 0$, so q is a critical point of $h|_M$.

(5) As $x_n(p+y) - x_n(p) = p_n + y_n - p_n = y_n$, where we write $p = (p_1, \ldots, p_n)$, $y = (y_1, \ldots, y_n)$, $Dh(p)y = y_n$ for any $p \in \mathbb{R}^n$ by the definition of the derivative. This is the zero map on T_pM if and only if $v \in T_pM$, $v = (v_1, \ldots, v_n)$, implies $v_n = 0$. But every vector v orthogonal to p is in T_pM (consider the curve given by $\gamma(t) = \frac{p+tv}{\|p+tv\|}$, so $\gamma'(0) = v - (v \cdot p)p = v$), so p needs to lie in the orthocomplement of the plane spanned by the first $n-1$ coordinate vectors, i.e. needs to be a multiple of $(0, \ldots, 0, 1)$; as $p \in M$, we deduce that the only critical points of $h|_M$ are $q_1 = (0, \ldots, 0, 1)$ and $q_2 = (0, \ldots, 0, -1)$. (Of course, one could have argued using ∇h instead.)

If p is one of these two points q_j, the flow is constant, so the $t \to \pm\infty$ limits of the flow are both the same q_j. Otherwise, as h is increasing, and $\lim_{t\to+\infty} \mathcal{F}_X^t(p)$ exists, and is a critical point of h, $\lim_{t\to+\infty} \mathcal{F}_X^t(p) = q_1$, and similarly $\lim_{t\to-\infty} \mathcal{F}_X^t(p) = q_2$.

Problem 3. *Suppose V is a finite dimensional real vector space, $T^*V = V \times V^*$ its cotangent bundle. For $p \in V$, let ω_p be the symplectic form on T_pT^*V, and let $J_p : T_pT^*V \to T_p^*T^*V$ be the isomorphism induced by ω_p as in Problem Set 4.*

(1) *If $e_1, \ldots, e_n$ is a basis of V, $f_1, \ldots, f_n$ is the dual basis of V^*, then*

$$(e_1, 0), \ldots, (e_n, 0), (0, f_1), \ldots, (0, f_n)$$

is a basis of $T^*V = V \times V^*$; one often writes the corresponding coordinates on T^*V as $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$. Find the matrix of J_p explicitly with respect to this basis.

(2) *Suppose that $f : T^*V \to \mathbb{R}$ is a C^∞ function. For $p \in T^*V$, define a vector $H_f(p) \in T_pT^*V$ by $H_f(p) = J_p^{-1} df(p)$, where $df(p) \in T_p^*T^*V$ is the differential of f at p. Show that*

$$\omega_p(v, H_f(p)) = df(p)v$$

for all $v \in T_pT^*V$, and $H_f : T^*V \to TT^*V$, $H_f : p \mapsto H_f(p)$, is a C^∞ map.

(3) *Find $H_f(p)$ explicitly in the basis described in (1).*

(4) *A bicharacteristic γ is an integral curve of H_f, i.e. a C^1 map $\gamma : I \to T^*V$ such that $\gamma'(t) = H_f(\gamma(t))$ for all $t \in I$. Write down this equation explicitly as an ODE using the basis described in (1).*

(5) *Show that bicharacteristics are C^∞. Show also that f is constant along bicharacteristics γ, i.e. $f(\gamma(t_1)) = f(\gamma(t_2))$ for all $t_1, t_2 \in I$.*

(6) *Suppose that f is such that $f(p) \to +\infty$ as $\|p\| \to \infty$ – here $\|.\|$ is some norm on T^*V (recall that all norms are equivalent on T^*V). Show that the*

integral curves γ can be defined on all of $\mathbb{R}$ (i.e. any integral curve can be extended to another one that is defined on all of $\mathbb{R}$).

(7) *Now suppose $V = \mathbb{R}^n$ with the standard inner product, $f((x,\xi)) = \frac{1}{2}(a\|x\|^2 + \|\xi\|^2)$, $a > 0$. Find the bicharacteristics explicitly.*

In classical mechanics, one regards V as the configuration space, usually corresponding to positions of particles, T^*V as the phase space, with the V^* factor corresponding to momenta, f is the energy function on T^*V, and bicharacteristics are trajectories of particles, and the corresponding ODE is Hamilton's equations of motion. The constancy of f along bicharacteristics is the conservation of energy. If $f(p) \to +\infty$ as $\|p\| \to \infty$, one calls f confining. The example is the harmonic oscillator; $\frac{1}{2}\|\xi\|^2$ is the kinetic energy and $\frac{a}{2}\|x\|^2$ is the potential energy.

Note that H_f and the bicharacteristics are defined without reference to a basis – so they do not depend on choices of a basis. In other words, the dynamics looks the same no matter what basis you choose. In fact, more is true: even diffeomorphisms 'preserve the dynamics', i.e. it is not affected by changes of coordinates – provided one relates f in the two settings to each other the correct way.

Solution. (1) We identify T_pT^*V with T^*V as usual. For $w = (x,\xi), w' = (y,\eta) \in T^*V$,

$$J(w)(w') = \Omega(w', w) = \eta(x) - \xi(y) = \sum_{j=1}^{n}(\eta_j x_j - \xi_j y_j),$$

In other words, $J(w)$ is the linear map the maps (y,η) to $\sum_{j=1}^n(-\xi_j y_j + x_j\eta_j)$. Now, $(e_k, 0)$ is the vector with $x_k = 1$, all other coordinates 0, so $J(e_k,0)(y,\eta) = \eta_k$, while $J(0,f_k)(y,\eta) = -y_k$. Now, $(T^*V)^* = (V \oplus V^*)^* = V^* \oplus V$, so a basis for it is $(f_j, 0)$, $(0, e_j)$, $j = 1,\ldots,n$, and our calculation says that

$$J(e_k,0)(y,\eta) = \eta_k = (0,e_k)(y,\eta),\ J(0,f_k)(y,\eta) = -y_k = (-f_k,0)(y,\eta),$$

where on the right hand side we evaluate the element $(0,e_k)$, resp. $(f_k,0)$ of $(T^*V)^*$ on (y,η). This means that using these bases of T^*V and $(T^*V)^*$, namely $(e_k,0)$, $(0,f_k)$ for T^*V, resp. $(f_k,0)$, $(0,e_k)$ for $(T^*V)^*$ (in this order!), the matrix of J is

$$\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.$$

(2) By definition of J, $\Omega(w',w) = J(w)w'$, so applying this with $w = J^{-1}u$, $\Omega(w', J^{-1}u) = v(w')$. With $u = df(p)$, this gives

$$\omega_p(v, H_f(p)) = df(p)v.$$

H_f is C^∞ as df is C^∞ since f is C^∞, and J is an invertible linear map, hence both J and J^{-1} are C^∞.

(3) Explicitly, the matrix of J^{-1} is

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

and

$$df(p) = \sum_j \frac{\partial f}{\partial x_j}(f_j,0) + \sum_j \frac{\partial f}{\partial \xi_j}(0,e_j),$$

so

$$H_f(p) = J_p^{-1} df(p) = \sum_j \frac{\partial f}{\partial x_j} J_p^{-1}(f_j, 0) + \sum_j \frac{\partial f}{\partial \xi_j} J_p^{-1}(0, e_j)$$
$$= -\sum_j \frac{\partial f}{\partial x_j}(0, f_j) + \sum_j \frac{\partial f}{\partial \xi_j}(e_j, 0).$$

(4) Writing $\gamma(t) = (x(t), \xi(t)) = \sum x_j(t)(e_j, 0) + \sum \xi_j(t)(0, f_j)$, we obtain the ODE
$$\frac{dx_j}{dt} = \frac{\partial f}{\partial \xi_j},\ \frac{d\xi_j}{dt} = -\frac{\partial f}{\partial x_j}.$$

(5) Since f is C^∞, H_f is C^∞, hence the bicharacteristics are C^∞. Also
$$(f \circ \gamma)'(t) = (df)(\gamma(t))\gamma'(t) = (df)(\gamma(t))H_f(\gamma(t))$$
$$= \omega_{\gamma(t)}(H_f(\gamma(t)), H_f(\gamma(t))) = 0$$
as Ω is antisymmetric, i.e. $\Omega(v, v') = -\Omega(v', v)$ so both sides vanish if $v = v'$.

(6) Since f is constant along bicharacteristics, if γ is defined on an interval I, $f \circ \gamma$ is constant on I, in particular $\gamma(t)$ does not go to infinity. (Because if it did, $f(\gamma(t))$ would too.) But as long as they are bounded, the solutions of an ODE on a vector space cannot break down, i.e. they can be extended to all of $\mathbb{R}$.

(7) If $f((x, \xi)) = \frac{1}{2}(a\|x\|^2 + \|\xi\|^2)$, the ODE's become
$$\frac{dx_j}{dt} = \xi_j,\ \frac{d\xi_j}{dt} = -ax_j.$$
Thus, $\frac{d^2x_j}{dt^2} = \frac{d\xi_j}{dt} = -ax_j$, so the solution is
$$x_j(t) = A_j \cos\sqrt{a}t + B_j \sin\sqrt{a}t,$$
for some constants A, B. Then
$$\xi_j(t) = \frac{dx_j}{dt} = -\sqrt{a}A_j \sin\sqrt{a}t + \sqrt{a}B_j \cos\sqrt{a}t,$$
so if we demand that $x(0) = x^0$, $\xi(0) = \xi^0$, we obtain $A_j = x_j^0$, $B_j = \xi_j^0/\sqrt{a}$, i.e.
$$x_j(t) = x_j^0 \cos\sqrt{a}t + \frac{\xi_j^0}{\sqrt{a}} \sin\sqrt{a}t,$$
$$\xi_j(t) = -\sqrt{a}x_j^0 \sin\sqrt{a}t + \xi_j^0 \cos\sqrt{a}t.$$
Note that $ax_j(t)^2 + \xi_j(t)^2 = a(x_j^0)^2 + (\xi_j^0)^2$, hence the energy f is conserved, confirming the correctness of our calculation.