Problem 1. Use separation of variables to solve the Dirichlet problem for the Laplacian: $\Delta u = 0$ in $\Omega$, $u|_{\partial \Omega} = f$ given, where $\Omega = \{ z \in \mathbb{R}^2 : a < |z| < b \}$, $a, b > 0$, is an annulus. (Hint: remember that there are two linearly independent solutions of the radial ODE. Keep them both. Also let $f_a$, resp. $f_b$ be $f$ at the two boundary circles, so $u(z) = f_a(z)$, if $|z| = a$, etc.)

Problem 2. The Fourier sine series of a function $f$ on $[0, \ell]$, $\ell > 0$, is the expansion

$$\sum_{n=1}^{\infty} a_n e_n, \quad e_n(x) = \sin(n\pi x/\ell), \quad a_n = (f, e_n)_{[0,\ell]}, \quad (f, g)_{[0,\ell]} = \frac{2}{\ell} \int_0^\ell f(x) g(x) dx.$$  

1. Show that if $f \in C^1([0, \ell])$ and satisfies homogeneous Dirichlet boundary conditions, i.e. $f(0) = 0$, $f(\ell) = 0$, then the Fourier sine series converges to $f$ uniformly.

2. Suppose $f$ is piecewise $C^1$ on $[0, \ell]$. What does its Fourier sine series converge to pointwise?

Hint: if $f \in C^1([0, \ell])$, extend $f$ to be an odd $2\ell$-periodic function $F$ on $\mathbb{R}$, let $g(x) = F(\pi x/\ell)$, and use the results from $S^1$. Notice that the Fourier sine series of $f$ just becomes the standard Fourier series of $g$!

Note: if $f \in L^2([0, \ell])$, then the Fourier sine series of $f$ converges to $f$ in $L^2$, but you do not need to prove this.

Problem 3.  
(1) Prove the following maximum principle for the heat equation $u_t = k\Delta u$ on $D = \Omega \times (0, T)$, $\Omega \subset \mathbb{R}^n$ bounded, open, $T > 0$, $k > 0$:

If $u \in C^2(D) \cap C(\overline{D})$, then $\max u$ is attained either on $\partial \Omega \times [0, T]$ or on $\overline{D} \times \{0\}$.

(2) Use this to state and prove a uniqueness and stability result for solutions of the heat equation: $u_t = k\Delta u$ on $\Omega \times (0, \infty)$, $u(x, t) = h(x, t)$ given if $x \in \partial \Omega$, $u(x, 0) = \phi(x)$ given, $x \in \Omega$.

Problem 4. Solve the heat equation on the interval, representing a rod whose ends are kept at temperature 0, and whose initial temperature is $\phi$:

$$u_t = k u_{xx}, \quad (x, t) \in (0, \ell) \times (0, \infty), \quad k > 0,$$

$$u(0, t) = 0, \quad u(\ell, t) = 0,$$

$$u(x, 0) = \phi(x),$$

$\phi \in C^1([0, \ell])$, $\phi(0) = 0 = \phi(\ell)$, by separating variables and using Fourier sine series in $x$. Make sure that you prove that your series solutions actually solves the PDE and satisfies the boundary and initial conditions.

Show also that the solution is $C^\infty$ for $t > 0$.

Problem 5. (Taylor 3.3.1) Let $C_0(\mathbb{R}^n)$ denote the space of continuous functions $v$ on $\mathbb{R}^n$ such that $v(\xi) \to 0$ as $|\xi| \to \infty$. Show that the Fourier transform satisfies $\mathcal{F} : L^1(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$. (Hint: $S(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$, and prove that if $f_j$ is a sequence in $S(\mathbb{R}^n)$ which converges uniformly, the limit $f$ is in $C_0(\mathbb{R}^n)$.) This is the Riemann-Lebesgue lemma.