Problem 1. (Cf. Taylor, I.1.4) Recall that a map $F$ between complex vector spaces is called differentiable (in the complex sense) at $x$ if there is a complex linear map $L$ such that $F(x + y) = F(x) + Ly + R(x, y)$ with $\lim_{\|y\|\to 0} \frac{\|R(x, y)\|}{\|y\|} = 0$. If $F$ is differentiable as a complex map, with $DF$ continuous, one calls $F$ holomorphic, or complex analytic.

Identify $\mathbb{C}^n$ with $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ by writing $z \in \mathbb{C}^n$ as $z = x + iy$, $x, y \in \mathbb{R}^n$.

Multiplication by $i$ on $\mathbb{C}^n$ becomes multiplication by a matrix $J_n$ on $\mathbb{R}^{2n}$,

$$J_n = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix},$$

where $I$ is the identity operator on $\mathbb{R}^n$.

Show that for $O \subset \mathbb{C}^n$ open, $F : O \to \mathbb{C}^m$ is holomorphic if and only if $F$ is $C^1$ as a real map (i.e. identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, $\mathbb{C}^m$ with $\mathbb{R}^{2m}$), and $DF \in \mathcal{L}(\mathbb{R}^{2n}, \mathbb{R}^{2m})$, regarded as a real linear map, satisfies

$$J_m DF(z) = DF(z) J_n$$

for all $z \in O$.

Write out $DF(z)$ as a block matrix corresponding to $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$, and similarly with $\mathbb{R}^{2m}$, writing $F = u + iv$, $u, v : O \to \mathbb{R}^m$, and deduce the Cauchy-Riemann equations

$$D_x u = D_y v, \quad D_y u = -D_x v$$

hold if and only if $F$ is holomorphic.

Problem 2. Show that all norms on a finite dimensional vector space $V$ are equivalent, i.e. if $\| \cdot \|_1$ and $\| \cdot \|_2$ are norms on $V$ then there is a constant $C > 0$ such that $\|v\|_1 \leq C \|v\|_2$ and $\|v\|_2 \leq C \|v\|_1$ for all $v \in V$. (Hint: You may assume $V$ is real as every complex vector space is also a real vector space. Use a basis to reduce the question to norms on $\mathbb{R}^n$. Now use the fact that the unit sphere in $\mathbb{R}^n$ is compact in the Euclidean topology, and show that all norms on $\mathbb{R}^n$ are equivalent to the Euclidean norm.)

Problem 3. Do Taylor I.2.3.

For this problem, take the following extrinsic definition of a smooth surface in $\mathbb{R}^n$: A smooth $k$-dimensional surface $M$ ($0 \leq k \leq n$) is a subset of $\mathbb{R}^n$, with the relative topology, such that for each $p \in M$ there is a neighborhood $O$ of $p$ in $\mathbb{R}^n$ and a smooth map: $\Phi : O \to \mathbb{R}^{n-k}$ such that

$$M \cap O = \Phi^{-1}\{0\} = \{x \in O : \Phi(x) = 0\}$$

and $D\Phi(p) : \mathbb{R}^n \to \mathbb{R}^k$ is a surjective linear map.

A smooth hypersurface is a smooth $n-1$-dimensional surface in $M$, in which case $\Phi$ is simply a real valued function, and the assumption is that $D\Phi(p) \neq 0$.

The meaning of smooth depends on the context, it usually means either $C^1$ or $C^\infty$. ($C^1$ suffices here.)
Also use the following definition of tangency. First, a curve in $M$ is a smooth map $\gamma : I \to \mathbb{R}^n$ and $\gamma(t) \in M$ for $t \in I$; here $I$ is an interval. A vector $V \in \mathbb{R}^n$ is tangent to $M$ at $p \in M$ if there is a curve $\gamma : I \to M$, $I$ a neighborhood of 0, $\gamma(0) = p$, such that $\gamma'(0) = V$. A vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ is tangent to $M$ if for all $x \in M$, $F(x)$ is tangent to $M$ at $x$.

In order to do the ‘straightening out’ of the hint, use the implicit function theorem by breaking up the standard coordinate functions on $\mathbb{R}^n$ into two groups (which you may then rearrange), $(x, z), x \in \mathbb{R}^k, z \in \mathbb{R}^{n-k}$, such that $D_z \Phi(p)$ is invertible, so on a neighborhood $O' \subset O$ of $p$, $M \cap O' = \{(x, f(x)) : x \in U\}, U \subset \mathbb{R}^k$ open, $f$ smooth.

Now define a vector field $\tilde{F} : U \to \mathbb{R}^k$ such that solutions of $x'(t) = \tilde{F}(x(t))$ lift to solve the original ODE, i.e. letting $y(t) = (x(t), f(x(t))), y'(t) = F(y(t))$.

**Problem 4.** Do Taylor I.3.1.

**Problem 5.** Do Taylor I.3.4.

**Problem 6.** Do Taylor I.4.3.

**Problem 7.** Do Taylor I.4.4.