Problem 1. (20 points)

(i) The integral of a non-negative extended real-valued measurable function $f$ on $\mathbb{R}^d$ is given by

$$\int_{\mathbb{R}^d} f(x) dx = \sup_g \int_{\mathbb{R}^d} g(x) dx$$

where the supremum is taken over all bounded functions $0 \leq g \leq f$ which are bounded and supported on a set of finite measure.

(ii) First suppose $\alpha < -n$, and define a strictly decreasing sequence $x_m \downarrow 0$. We define step functions $g_m$ as follows. Define the sets $A_m = B_{x_m}(0) \setminus B_{x_m/2}(0)$, and set $g_m = x_m^\alpha \chi_{A_m}$. When $\alpha < -n$, we have $g_m(x) \leq f(x) := |x|^n \chi_{B_1(0)}$.

Evaluating the integral of $g_m$ yields

$$\int_{\mathbb{R}^n} g_m = x_m^\alpha m(A_m) = x_m^\alpha (\omega_n x_m^n - \omega_n (x_m/2)^n) = x_m^{\alpha+n} \omega_n \left( 1 - \left( \frac{1}{2} \right)^n \right).$$

But this expression is unbounded as $m \to \infty$ since $\alpha < -n$, so by definition we must have $\int_{\mathbb{R}^n} f = \infty$. To handle the case where $\alpha = -n$, consider the construction above with $x_m = 2^{-m}$. From the formula above, we see that the integral of each $g_m$ is the constant value $\omega_n (2^n - 1)/2^n$. However, if we define new step functions

$$G_m(x) = \sum_{\ell=0}^m g_\ell(x),$$

it is clear that $G_m(x) \leq f(x)$ for all $m$ but clearly $\int_{\mathbb{R}^n} G_m \to \infty$ as $m \to \infty$. So again we must have $\int_{\mathbb{R}^n} f = \infty$.

Next, consider the case where $\alpha > -n$. Suppose also that $\alpha < 0$, since otherwise the function is trivially dominated by $\chi_{B_1(0)}$. In this case we define functions $h_m = x_m^\alpha \chi_{B_{2x_m}(0)} \chi_{B_{x_m}(0)}$, and further set

$$H(x) = \sum_{m=0}^\infty h_m(x).$$

Now letting $x_m = 2^{-m}$, we see that $f(x) \leq H(x)$, and $H$ is integrable since

$$\int_{\mathbb{R}^n} H = \sum_{m=0}^\infty \int_{\mathbb{R}^n} h_m = \sum_{m=0}^\infty \omega_n (2^n - 1) 2^{-m(\alpha+n)},$$

and this is a geometric series since $\alpha > -n$. This proves that $f$ is integrable, so we are done.
(iii) We can express this integral in terms of polar coordinates. Letting $\omega_n$ denote the volume of the unit ball in $\mathbb{R}^n$, one easily checks that

$\int_{\mathbb{R}^n} |x|^\alpha \chi_{B_1(0)} \, dx = n\omega_n \int_0^1 r^{\alpha+n-1} \, dr. \tag{0.1}$

Note that one can deduce this formula from the fact that

$\omega_n = \int_0^{\pi} \cdots \int_0^{\pi} r^{n-1} \phi(\theta_1, \ldots, \theta_{n-1}) \, d\theta_1 \cdots d\theta_{n-1}$

$= \left( \int_0^{\pi} \cdots \int_0^{2\pi} \phi(\theta_1, \ldots, \theta_{n-1}) \, d\theta_1 \cdots d\theta_{n-1} \right) \left( \int_0^1 r^{n-1} \, dr \right)$

$= \frac{1}{n} \int_0^{\pi} \cdots \int_0^{2\pi} \phi(\theta_1, \ldots, \theta_{n-1}) \, d\theta_1 \cdots d\theta_{n-1}.$

When $\alpha > -n$, we evaluate the integral (0.1) to find

$\int_{\mathbb{R}^n} |x|^\alpha \chi_{B_1(0)} \, dx = \frac{n\omega_n}{\alpha + n}.$

**Problem 2. (15 points)**

We know that the $L^2([0, \pi])$-error is minimized when the coefficients are taken to be the generalized Fourier coefficients. In particular, this means means the the difference $\|f - \phi\|_{L^2}$ is minimized when

$b_1 = \frac{\langle \phi, \sin x \rangle_{L^2}}{\| \sin x \|_{L^2}} = \frac{\int_0^\pi x(\pi - x) \sin x \, dx}{\int_0^\pi \sin^2 x \, dx} = \frac{8}{\pi}$

$b_2 = \frac{\langle \phi, \sin 2x \rangle_{L^2}}{\| \sin 2x \|_{L^2}} = \frac{\int_0^{\pi} x(\pi - x) \sin 2x \, dx}{\int_0^{\pi} \sin^2 2x \, dx} = 0$

$b_4 = \frac{\langle \phi, \sin 4x \rangle_{L^2}}{\| \sin 4x \|_{L^2}} = \frac{\int_0^{\pi} x(\pi - x) \sin 4x \, dx}{\int_0^{\pi} \sin^2 4x \, dx} = 0$

$b_8 = \frac{\langle \phi, \sin 8x \rangle_{L^2}}{\| \sin 8x \|_{L^2}} = \frac{\int_0^{\pi} x(\pi - x) \sin 8x \, dx}{\int_0^{\pi} \sin^2 8x \, dx} = 0$

**Problem 3. (20 points)** Suppose $|K(x, y)| \leq C$ for some constant $C$. Since $K$ is a measurable function in $(x, y)$, by Fubini’s theorem, the slice function $K^x(y)$ is a measurable function in $y$ for almost every $x$. And for every such $x$, we have

$|g(x)| = \left| \int K(x, y) f(y) \, dy \right| \leq \int C |f(y)| \, dy = C \| f \|_{L^1}.$

So $g(x)$ is a well-defined real valued function for almost every $x \in [0, 1]$. To prove the second statement, we just use the above inequality again:

$\|g\|_{L^1} = \int_0^1 |g(x)| \, dx \leq \int_0^1 C \| f \|_{L^1} = C \| f \|_{L^1}.$

That where the constant $C$ only depends on $K$. 
Problem 4. (30 points)

(i) Fubini’s Theorem: Suppose \( f(x, y) \) is integrable on \( \mathbb{R}^d_1 \times \mathbb{R}^d_2 \). Then for almost every \( y \in \mathbb{R}^{d_2} \), we have:

- The slice \( f^y \) is integrable on \( \mathbb{R}^{d_1} \).
- The function defined by \( \int_{\mathbb{R}^{d_1}} f^y(x) dx \) is integrable on \( \mathbb{R}^{d_2} \).
- We have \( \int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} f(x, y) dx \right) dy = \int_{\mathbb{R}^{d}} f \).

Tonelli’s Theorem: Suppose \( f(x, y) \) is a non-negative measurable function on \( \mathbb{R}^d_1 \times \mathbb{R}^d_2 \). Then for almost every \( y \in \mathbb{R}^{d_2} \):

- The slice \( f^x \) is measurable on \( \mathbb{R}^{d_2} \).
- The function defined by \( \int_{\mathbb{R}^{d_2}} f^x(y) dy \) is measurable on \( \mathbb{R}^{d_2} \).
- We have \( \int_{\mathbb{R}^{d_1}} \left( \int_{\mathbb{R}^{d_2}} f(x, y) dy \right) dx = \int_{\mathbb{R}^{d}} f(x, y) dy dx \) in the extended sense.

(ii) Since \( f, g \) are integrable, we know from the text that the functions \( (x, y) \mapsto f(x - y) \) and \( (x, y) \mapsto g(y) \) are measurable, and therefore so is their product \( F(x, y) = f(x - y)g(y) \). This function is integrable, for an application of Tonelli’s theorem gives

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} |F| = \int_{\mathbb{R}^d \times \mathbb{R}^d} |f(x - y)||g(y)||dy|dx = \int_{\mathbb{R}^d} |g(y)| \left( \int_{\mathbb{R}^d} |f(x - y)| |dx| \right) dy = \|f\|_{L^1} \int_{\mathbb{R}^d} |g(y)| dy = \|f\|_{L^1} \|g\|_{L^1} < \infty.
\]

The function \( F \) is thus integrable, so Fubini’s theorem implies that \( F^x \) is integrable for almost every \( x \) as well. Hence

\[
(f \ast g)(x) = \int_{\mathbb{R}^d} F^x(y) dy
\]

is defined for a.e. \( x \).

(iii) From the triangle inequality, we have

\[
\int_{\mathbb{R}^d} |f \ast g|(x) dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F^x(y) dy dx \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |F|
\]

which, by part (ii), is bounded above by \( \|f\|_{L^1} \|g\|_{L^1} \). Clearly the “\( \leq \)” sign can be replaced by an “\( = \)” sign when \( f, g \) are both non-negative, since in that case \( F \geq 0 \).

Problem 5. (20 points)
(i) First observe that since \( \hat{d}_k \leq 1 \) for all \( k \), \( d(f, g) \) is finite for all \( f, g \in C^\infty(S^1) \). Next there are three things to check. The first is that if \( d(f, g) = 0 \), then \( f = g \). But this is clear, for \( d(f, g) = 0 \) implies \( d_0(f, g) = 0 \), or \( |f(x) - g(x)| = 0 \) for all \( x \in S^1 \). The second is that \( d(f, g) = d(g, f) \) for all \( f, g \in C^\infty(S^1) \), which is also clear because each \( \hat{d}_k \) obviously satisfies this property. The third and final property of \( d \) we must check is that it satisfies the triangle inequality. Given \( f, g, h \in C^\infty(S^1) \), observe that
\[
\sup_{x \in S^1} |\partial_x f - \partial_x h| \leq \sup_{x \in S^1} |\partial_x f - \partial_x g| + \sup_{x \in S^1} |\partial_x g - \partial_x h|
\]
for each \( j \). It follows from this that each \( \hat{d}_k \) satisfies the triangle inequality for all \( k \), that is
\[
d_k(f, h) \leq d_k(f, g) + d_k(g, h).
\]
Now if \( d_k(f, h) < 1 \), obviously this statement will hold if either term on the right is replaced by a “1”. On the other hand, if \( d_k(f, h) \geq 1 \), we have
\[
1 = \hat{d}_k(f, h) < d_k(f, h) \leq d_k(f, g) + d_k(g, h)
\]
and, again, this statement obviously holds if either term on the right is replaced by a “1”. Putting these facts together allows us to conclude that
\[
\hat{d}_k(f, h) \leq d_k(f, g) + \hat{d}_k(g, h).
\]
Since each \( \hat{d}_k \) satisfies the triangle inequality, so too does \( d \), and therefore \((C^\infty(S^1), d)\) is a metric space.

(ii) Choose any \( \epsilon > 0 \), and first suppose that \( f_j \to f \) in \( C^\infty(S^1) \). This means that for some \( j_0 \), we have \( d(f, f_j) < 2^{-k_1} \epsilon \) for all \( j \geq j_0 \). But this implies \( \hat{d}_k(f, f_j) < \epsilon \Rightarrow d_k(f, f_j) < \epsilon \) (assuming we have chosen \( \epsilon < 1 \)), and therefore \( f_j \to f \) in \( C^k(S^1) \) for all \( k \).

For the other direction, suppose \( d_k(f, f_j) \to 0 \) (and hence \( \hat{d}_k(f, f_j) \to 0 \)) for all \( k \). Choosing \( K \) so large the \( \sum_{k=0}^{\infty} 2^{-k} < \epsilon/2 \), we can thus find a \( j_0 \) such that \( \hat{d}_k(f, f_j) < \epsilon/4 \) for \( 1 \leq k \leq K - 1 \). Therefore
\[
d(f, f_j) = \sum_{k=0}^{K-1} 2^{-k} \hat{d}_k(f, f_j) + \sum_{k=K}^{\infty} 2^{-k} \hat{d}_k(f, f_j) \leq \sum_{k=0}^{K-1} 2^{-k} + \frac{\epsilon}{2} \leq \epsilon,
\]
proving that \( f_j \to f \) in \( C^\infty(S^1) \).

Problem 6. (20 points)

(i) Since \( U(t) = K_t * \phi \), taking the Fourier transform yields \( \hat{U}(t) = \hat{K_t} \hat{\phi} \). Since the Fourier transform preserves \( L^2 \)-norms, to prove the claim we need only show that \( \| \hat{U}(t) - \hat{\phi} \|_{L^2} = \| \hat{\phi(K_t - 1)} \|_{L^2} \to 0 \) as \( t \to 0 \). To show this, one may use the given formula (in the instructions) to show that \( K_t(\xi) = e^{-kt\xi^2} \). Therefore
\[
\| \hat{\phi(K_t - 1)} \|_{L^2}^2 = \int \mathbb{R} (1 - e^{-kt\xi^2})^2|\hat{\phi}(\xi)|^2 d\xi.
\]
Now the integrand clearly approaches 0 pointwise as \( t \to 0 \), and it is also dominated by the integrable function \( |\hat{\phi}(\xi)|^2 \), so the dominated convergence
theorem gives that this integral (and hence \( \|U(t) - \phi\|_{L^2} \)) converges to 0 as \( t \to 0 \). Now to show that this solution \( u \) is smooth, note that since \( \hat{K}_t \) is Schwartz for all \( t > 0 \), it follows easily that \((-i\xi)^m \hat{K}_t \) belongs to \( L^2(\mathbb{R}_\xi) \) for all \( m \geq 0 \). We can thus show that \((-i\xi)^m \hat{K}_t \hat{\phi} \) belongs to \( L^1(\mathbb{R}_\xi) \), for the Cauchy-Schwartz inequality gives

\[
\int_{\mathbb{R}} \left| (-i\xi)^m \hat{K}_t \hat{\phi} \right| \leq \left( \int_{\mathbb{R}} \left| (-i\xi)^m \hat{K}_t \right|^2 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} \left| \hat{\phi} \right|^2 \right)^{\frac{1}{2}} < \infty
\]

since \( \phi \in L^2 \). Hence since \((-i\xi) \hat{K}_t \hat{\phi} \in L^1 \), the inverse transform of this function must be bounded and continuous. But this function is precisely \( \partial_x^n(K_t \ast \phi) \), and so all \( x \)-derivatives of \( K_t \ast \phi \) are bounded and continuous. More generally, the Fourier transform of the distributional derivative \( \partial_x^n \hat{\phi} \) is given by \((-i\xi)^m \partial_x^n \hat{K}_t \hat{\phi} \), and by the same reasoning as above each of these functions belongs to \( L^1(\mathbb{R}_\xi) \) locally uniformly in time (i.e. the \( L^1(\mathbb{R}_\xi) \)-norm of this function is bounded in any sufficiently small neighborhood of some \( t > 0 \)). Hence the inverse Fourier transform of each of these functions is bounded and continuous, and therefore each distributional derivative of \( K_t \ast \phi \) is bounded and continuous (locally uniformly in \( t \)). Therefore \( u \in C^\infty \).

(ii) Again using the fact that the Fourier transform preserves \( L^2 \)-norms, we have

\[
E'(t) = \partial_t \|U(t)\|_{L^2}^2 = \partial_t \|\hat{U}(t)\|_{L^2}^2 = \partial_t \int_{\mathbb{R}} \hat{K}_t^2 \hat{\phi}^2.
\]

Using the fact that \( \hat{K}_t \) is Schwartz for all \( t > 0 \), we may differentiate under the integral sign to find that this expression is equal to

\[
\int_{\mathbb{R}} \partial_t (\hat{K}_t^2) \hat{\phi}^2 = -2k \int_{\mathbb{R}} \xi^2 \hat{K}_t^2 \hat{\phi}^2 \leq 0,
\]

proving that \( E(t) \) is monotone decreasing.

**Problem 7.** (25 points)

(i) A function \( f \) belongs to \( S(\mathbb{R}^n) \) if \( f \) is smooth and for each \( N \geq 0 \) and each multiindex \( \alpha \in \mathbb{N}^n \), the function \( |x|^N D^\alpha f \) is bounded in \( \mathbb{R}^n \).

(ii) Denote the Fourier transform (in \( x \)) of \( u \) by \( \hat{u} = \mathcal{F}_x u \). Taking this transform of the given equation, we have

\[
\partial_t \hat{u} = -|\xi|^4 \hat{u}, \quad \hat{u}|_{t=0} = \hat{\phi}.
\]

Solving this ODE for \( \hat{u} \) gives \( \hat{u}(\xi, t) = e^{-|\xi|^4 t} \hat{\phi} \). Note that since \( \hat{\phi} \) is Schwartz, the given function is smooth for any \( t > 0 \). Moreover, \( \hat{u}(\xi, t) \) is Schwartz locally uniformly in \( t \) in that \( |\xi|^N D^\alpha \hat{u}(\cdot, t) \) is uniformly bounded in any sufficiently small neighborhood of \( t \) for all \( N > 0 \) and every multiindex \( \alpha \in \mathbb{N}^n \). Moreover, one easily checks that the same is true for all \( t \)-derivatives of \( \hat{u} \), so we conclude that

\[
u(x, t) = \mathcal{F}_x^{-1} \left( e^{-|\xi|^4 t} \hat{\phi} \right)
\]
is a $C^2$ solution of the given equation.

(iii) When $n = 1$ and $\phi(x) = e^{-x^2}$, we know that the Fourier transform is given by

$$\hat{\phi}(\xi) = \sqrt{\pi}e^{-\xi^2/4}.$$  

This is clearly a Schwartz function, and so, plugging this into our formula from part (ii) gives us

$$u(x, t) = \mathcal{F}_\xi^{-1} \left( \sqrt{\pi}e^{-t\xi^2-\xi^2/4} \right)$$

Problem 8. (35 points)

(i) Observe that the initial conditions suggest that, if we extend $\phi$ and $\psi$ to an odd, $2\ell$-periodic function on $\mathbb{R}$, this extension is $C^2$ since their second derivatives vanish at the points $n\ell$. Similarly, since the fourth derivatives also vanish at these points, we get that the second derivatives are $C^2$, and so on up to $k + 1$ derivatives. It follows, then, that these extensions of $\phi$ and $\psi$ converge to their respective sine series

$$\phi(x) = \sum_{n \neq 0} \phi_n \sin \left( \frac{n\pi x}{\ell} \right), \quad \psi(x) = \sum_{n \neq 0} \psi_n \sin \left( \frac{n\pi x}{\ell} \right),$$

and this convergence is uniform in $C^{k-1}$. Clearly we also have this uniform convergence of the interval $[0, \ell]$. The standard separation of variables argument leads us to conclude that

$$u(x, t) = \sum_{n \neq 0} \sin \left( \frac{n\pi x}{\ell} \right) \left[ \phi_n \cos \left( \frac{n\pi ct}{\ell} \right) + \frac{\ell}{n\pi c} \psi_n \sin \left( \frac{n\pi ct}{\ell} \right) \right].$$

To show that this series converges to a function which is $C^k$, observe that since $\phi \in C^{k+1}$, we know that there is some constant $C > 0$ such that $|\psi_n| \leq C/|n|^{k+1}$ and there is therefore some $C' > 0$ such that $|\xi_n| \leq C'/|n|^{k+2}$. It follows, therefore, that if we set

$$u_\psi = \sum_{n \neq 0} \sin \left( \frac{n\pi x}{\ell} \right) \frac{\ell}{n\pi c} \psi_n \sin \left( \frac{n\pi ct}{\ell} \right),$$

we find that the corresponding Fourier coefficients for the function $\partial_x^\alpha \partial_t^\beta u_\psi$, where $\alpha + \beta \leq k$, are uniformly bounded above by $C''/|n|^2$ for some $C'' > 0$. It thus follows from the Weierstrass $M$-test that this series converges uniformly, and so $\partial_x^\alpha \partial_t^\beta u_\psi$ must be continuous. It follows from this that $u_\psi \in C^k$, and hence $u \in C^k$.

(ii) Suppose that $u^{(1)}$ and $u^{(2)}$ are two $C^3$ solutions of the given initial value problem, and set $u = u^{(1)} - u^{(2)}$. Since $u \in C^3$, we know that $u$ is equal to its Fourier series in $x$ with coefficients depending on $t$, or

$$u(x, t) = \sum_{n \neq 0} A_n(t) \sin \left( \frac{n\pi x}{\ell} \right).$$
But if we plug this formula into the wave equation, we see that each function \( A_n(t) \) must satisfy the ODE
\[
A_n''(t) = -\frac{n^2 \pi^2 c^2}{\ell^2} A_n(t) \\
A_n'(0) = 0 \\
A_n(0) = 0.
\]
However, we know from ODE uniqueness that \( A_n(t) = 0 \) is the only solution to this boundary value problem, so we conclude that \( u \equiv 0 \), or \( u^{(1)} = u^{(2)} \).

(iii) The given initial conditions imply that \( \phi_n = 0 \) for all \( n \), while
\[
\psi_3 = 1, \quad \psi_1 = -2
\]
and \( \psi_n = 0 \) for all other \( n \). Plugging these values into our series for \( u \) gives us
\[
u(x, t) = -\frac{2\ell}{\pi c} \sin \left( \frac{\pi x}{\ell} \right) \sin \left( \frac{\pi c t}{\ell} \right) + \frac{\ell}{3\pi c} \sin \left( \frac{3\pi x}{\ell} \right) \sin \left( \frac{3\pi c t}{\ell} \right).
\]