Math 172
Problem Set 6 Solutions

Problem 1 4.5a As suggested in the hint, we first consider a function $f(x) = |x|^{-\alpha}$ when $|x| \leq 1$, and zero everywhere else. Using polar coordinates, and letting $\omega_d$ denote the area of the unit sphere in $\mathbb{R}^d$, we compute

$$\|f\|^2_{L^2} = \int_{\mathbb{R}^d} |f|^2 = \omega_d \int_0^1 r^{d-1-2\alpha} dr,$$

which is finite if and only if $d - 1 - 2\alpha > -1 \iff \alpha < d/2$. A similar calculation reveals that $\|f\|_{L^1} = \omega_d \int_0^1 r^{d-1-\alpha} dr$, which is finite if and only if $\alpha < d$. Hence if we choose $d/2 < \alpha < d$, we find that $\|f\|_{L^1} < \infty$ while $\|f\|_{L^2} = \infty$, proving that $L^1(\mathbb{R}^d) \nsubseteq L^2(\mathbb{R}^d)$. If we define $g(x) = |x|^{-\alpha}$ when $|x| > 1$ and zero everywhere else, a virtually identical argument shows that, for $\alpha$ chosen in the same range as above, $\|g\|_{L^2} < \infty$ while $\|g\|_{L^1} = \infty$. We thus also conclude that $L^2(\mathbb{R}^d) \nsubseteq L^1(\mathbb{R}^d)$.

4.5b When $f$ is supported in a set $E$ of finite measure, using the Cauchy-Schwarz inequality we have

$$\|f\|_{L^1} = \int_{\mathbb{R}^d} |f| = \int_{\mathbb{R}^d} |f| \chi_E \leq \left( \int_{\mathbb{R}^d} |f|^2 \right)^{1/2} \left( \int_{\mathbb{R}^d} |\chi_E|^2 \right)^{1/2} = m(E)^{1/2} \|f\|_{L^2},$$

as claimed.

4.5c Of course when $|f| \leq M$ we have

$$\|f\|_{L^2} = \left( \int_{\mathbb{R}^d} |f|^2 \right)^{1/2} \leq \left( \int_{\mathbb{R}^d} M^2 \right)^{1/2} = M^{1/2} \|f\|_{L^1}^{1/2},$$

as claimed.

Problem 2 4.6a Let $f \in L^2(\mathbb{R}^d)$. First assume that $f$ is real-valued, and write $f = f_+ - f_-$ as usual. There are non-negative simple functions $s_i \nearrow f_+$ and $r_i \nearrow f_-$, so if we define new simple functions $f_i = s_i - r_i$, we find that $|f - f_i|^2 \to 0$ pointwise. On the other hand, $|f - f_i|^2$ is dominated by the
integrable function \(|f|^2\). Hence the dominated convergence theorem implies that \(f_i \rightarrow f\) in \(L^2\), so we conclude that simple functions are dense in \(L^2\). If \(f\) is complex-valued, simply decompose \(f\) into its real and imaginary parts and apply the argument above to each part separately.

**4.6b** By part a, we need only show that for any positive simple function \(s\), we can find a sequence of compactly supported continuous functions \(h_i\) such that \(\|s - h_i\|_{L^2} \rightarrow 0\). Again, we assume that \(s\) is real-valued, as our argument extends trivially to the complex-valued case. Since all simple functions are bounded, there is some number \(M\) such that \(s \leq M\). Moreover, we know that we can find a sequence \(h_i\) such that \(\|s - h_i\|_{L^1} \rightarrow 0\). Without loss of generality, we may assume that \(|h_i| \leq M\) as well (for we can replace \(h_i\) with \(\min(h_i, M)\) where necessary). It then follows from exercise 4.5 above that we may bound \(\|s - h_i\|_{L^2}\) by

\[
\|s - h_i\|_{L^2} \leq (2M)^{1/2}\|s - h_i\|_{L^1} \rightarrow 0,
\]

proving the assertion.

**Problem 3 4.7** We first show that the set \(\{\varphi_{k,j}\}\) is orthonormal. To do this, we claim that for any \(1 \leq k, j, k', j' < \infty\) the product \(\varphi_{k,j} \overline{\varphi_{k',j'}}\) is integrable. Indeed we have

\[
\int_{\mathbb{R}^d} \left| \varphi_k(x) \varphi_j(y) \overline{\varphi_{k'}(x)} \overline{\varphi_{j'}(y)} \right| dx dy \\
\leq \left( \int_{\mathbb{R}^d} \left| \varphi_k(x) \varphi_j(y) \right|^2 dx dy \right)^{1/2} \left( \int_{\mathbb{R}^d} \left| \varphi_{k'}(x) \varphi_{j'}(y) \right|^2 dx dy \right)^{1/2} \\
= \left[ \left( \int_{\mathbb{R}^d} \left| \varphi_k(x) \right|^2 dx \right) \left( \int_{\mathbb{R}^d} \left| \varphi_j(y) \right|^2 dy \right) \right]^{1/2} \left[ \left( \int_{\mathbb{R}^d} \left| \varphi_{k'}(x) \right|^2 dx \right) \left( \int_{\mathbb{R}^d} \left| \varphi_{j'}(y) \right|^2 dy \right) \right]^{1/2},
\]

where we have used first the Cauchy-Schwarz inequality and then Tonelli’s theorem. Since this final expression is equal to 1, the product \(\varphi_{k,j} \overline{\varphi_{k',j'}}\) is integrable in \(\mathbb{R}^d \times \mathbb{R}^d\). This observation allows us to apply Fubini’s theorem to prove that

\[
(\varphi_{k,j}, \varphi_{k',j'})_{L^2} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi_k(x) \varphi_j(y) \overline{\varphi_{k'}(x)} \overline{\varphi_{j'}(y)} dx dy \\
= \left( \int_{\mathbb{R}^d} \varphi_k(x) \overline{\varphi_{k'}(x)} dx \right) \left( \int_{\mathbb{R}^d} \varphi_j(y) \overline{\varphi_{j'}(y)} dy \right) = \begin{cases} 1 & \text{if } (k,j) = (k',j') \\ 0 & \text{otherwise} \end{cases}.
\]

Hence the set \(\{\varphi_{k,j}\}\) is orthonormal in \(L^2(\mathbb{R}^d \times \mathbb{R}^d)\). To prove that this set is a basis, by Theorem 4.2.3 in the text it suffices to show that, given any \(F \in L^2\), if \((F, \varphi_{k,j})_{L^2} = 0\) for all \(k, j\), then \(F \equiv 0\). We define functions \(F_j\) as in the hint.
We observe that \((F, \varphi_{k,j})_{L^2} = 0\) implies
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} F(x,y)\overline{\varphi_k(x)}\overline{\varphi_j(y)}dxdy = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} F(x,y)\overline{\varphi_j(y)}dy\right)\overline{\varphi_k(x)}dx
= \int_{\mathbb{R}^d} F_j(x)\overline{\varphi_k(x)}dx = (F_j, \varphi_k)_{L^2} = 0.
\]
Since this holds for all \(k\), we must have \(F_j(x) = 0\) for all \(j\). But this implies that for all \(x \in \mathbb{R}^d\),
\[
0 = F_j(x) = \int_{\mathbb{R}^d} F(x,y)\overline{\varphi_j(y)}dy
\]
for all \(j\). Since an easy application of Fubini’s theorem shows that \(F^x \in L^2(\mathbb{R}^d)\) for a.e. \(x\), we conclude that \(F^x = 0\) identically for a.e. \(x\), and hence \(F = 0\) in \(L^2\).

**Problem 4 4.8a** We check that \(L^2_\eta := L^2([a,b], \eta)\) satisfies all the properties of a Hilbert space:

(i) Since \(\eta\) is strictly positive and continuous on the compact set \([a,b]\), we see that there are positive constants \(m, M\) such that \(m \leq \eta \leq M\), and hence
\[
m\|f\|_{L^2} \leq \|f\|_{L^2_\eta} \leq M\|f\|_{L^2}.
\]
It follows from this that \(L^2 = L^2_\eta\) as sets, and therefore \(L^2_\eta\) is a vector space.

(ii-iii) Clearly \(f \mapsto (f, g)_{L^2_\eta}\) is linear for any fixed \(g\), and \((f, g)_{L^2_\eta} = \overline{(g, f)}_{L^2_\eta}\) since \(\eta\) is real-valued. Moreover, from part (i) we see that \((f, f)_{L^2_\eta} \geq m(f, f)_{L^2} \geq 0\), and in particular \((f, f)_{L^2_\eta} = 0\) if and only if \(f = 0\).

(iv) We observe that
\[
|(f, g)_{L^2_\eta}| = \left|\int (f\eta^{1/2})(g\eta^{1/2})d\eta\right| \leq \left(\int |f|^2\eta\right)^{1/2} \left(\int |g|^2\eta\right)^{1/2}
\]
which proves the Cauchy-Schwarz inequality on \(L^2_\eta\), where we have made use of the Cauchy-Schwarz inequality on \(L^2\). Next, since \(\|f\|_{L^2_\eta} = \|f\eta^{1/2}\|_{L^2}\), we have
\[
\|f + g\|_{L^2_\eta} = \|f\eta^{1/2} + g\eta^{1/2}\|_{L^2} \leq \|f\eta^{1/2}\|_{L^2} + \|g\eta^{1/2}\|_{L^2} = \|f\|_{L^2_\eta} + \|g\|_{L^2_\eta},
\]
so the triangle inequality holds as well.

(v) To show that \(L^2_\eta\) is complete, suppose \(\{f_i\}\) is a Cauchy sequence in \(L^2_\eta\). Then \(\{f_i\eta^{1/2}\}\) is a Cauchy sequence in \(L^2\), and so by the completeness of this space, there is some function \(f\eta^{1/2}\) such that \(f_i\eta^{1/2} \to f\eta^{1/2}\) in \(L^2\). But this clearly implies that \(f_i \to f\) in \(L^2_\eta\), so the latter is also complete.
(vi) \(L^2\) is separable, and so there exists a basis \(\{\varphi_i\}\). It is clear that \(\varphi, \eta^{-1}\) is a basis for \(L^2_\eta\), so we conclude that \(L^2_\eta\) is separable.

Now consider the map \(U : L^2_\eta \to L^2\) given by \(Uf = \eta^{1/2}f\). We have that
\[
\|Uf\|_{L^2} = \int |f|^2 \eta = \|f\|_{L^2_\eta}
\]
and so \(U\) is an isometry. Moreover, it is clearly invertible since \(\eta\) is bounded away from zero. Hence \(U\) gives a unitary correspondence.

**Problem 5.** To make \(f\) closest to \(\phi\) in \(L^2\) sense, we just take the corresponding coefficients to be the Fourier coefficients with respect to \(L^2\) basis (not necessarily normal) \(\{1, \sin(x), \cos(x), \sin(2x), \cos(2x), \ldots\}\) on \([-\pi, \pi]\). We do the following calculation:
\[
\int_{-\pi}^{\pi} 1 = 2\pi, \quad \int_{-\pi}^{\pi} |x| dx = \pi^2,
\]
\[
\int_{-\pi}^{\pi} |x| \cos x dx = -4, \quad \int_{-\pi}^{\pi} \cos^2(x) dx = \pi,
\]
\[
\int_{-\pi}^{\pi} |x| \cos(2x) dx = 0, \quad \int_{-\pi}^{\pi} \cos^2(2x) dx = \pi.
\]

And since \(\phi(x) = |x|\) is even function on \([-\pi, \pi]\), its sine Fourier coefficients are all 0. So we conclude
\[
a_0 = \frac{\pi}{2}, a_1 = -\frac{4}{\pi}, a_2 = 0, b_1 = b_2 = 0.
\]

**Problem 6.** We use Zorn’s lemma, which can be shown to be equivalent to the Axiom of Choice. Recall Zorn’s lemma states that every partially ordered set contains a maximal element. In other words, if \((X, \leq)\) is a partially ordered set, there exists some element \(m\) such that \(m \leq x \Rightarrow m = x\). In our case, let \(X\) be the set of all linearly independent subsets of the Hilbert space \(H\), and we partially order this set by inclusion (i.e. if \(S_1, S_2\) are linearly independent sets and \(S_1 \subset S_2\) we set \(S_1 \leq S_2\)). By Zorn’s lemma, this set has a maximal element \(S = \{e_\alpha\}\). It is easy to see that this maximal linearly independent set is an algebraic basis, for if there were some \(\hat{e} \in H\) which were not expressible as a finite linear combination of the \(e_\alpha\)’s, then we could form a new linearly independent set \(\{\hat{e}\} \cup S \supset S\), which contradicts the maximality of \(S\). Now select (using induction and the axiom of choice, for example) a denumerable subset \(\{e_n\} \subset \{e_\alpha\}\), and define a functional \(\ell\) on \(H\) such that \(\ell(e_n) = n\) and \(\ell(e_\alpha) = 0\) if \(e_\alpha \notin \{e_n\}\). Since \(\ell\) is defined on an algebraic basis, it extends linearly to all of \(H\). Since it is clearly not bounded, we are done.

**Problem 7.**
(i) The Fourier sine series for the given function is \( \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x}{\ell} \right) \), where \( B_n \) is given by
\[
B_n = \frac{2}{\ell} \int_{0}^{\ell} x(\ell - x) \sin \left( \frac{n\pi x}{\ell} \right) \, dx = \begin{cases} 
0 & \text{if } n \text{ even} \\
\frac{8\ell^2}{n^2 \pi^2} & \text{if } n \text{ odd}
\end{cases}
\]
One may thus write the Fourier sine series for the given function as
\[ \sum_{k=0}^{\infty} \frac{8\ell^2}{(2k+1)^3 \pi^3} \sin \left( \frac{(2k+1)\pi x}{\ell} \right). \]
Since the given function is clearly in \( L^2 \), this series converges to the given function on \([0, \ell]\), and to its odd and \(2\ell\)-periodic extension on all of \( \mathbb{R} \).

(ii) The Fourier cosine series for the given function is given by
\[ A_0 + \sum_{n=1}^{\infty} A_n \cos \left( \frac{n\pi x}{\ell} \right), \]
where \( A_0 \) is given by
\[ A_0 = \frac{1}{\ell} \int_{0}^{\ell} x(\ell - x) \, dx = \frac{\ell^2}{6} \]
and the other coefficients are given by
\[ A_n = \frac{2}{\ell} \int_{0}^{\ell} x(\ell - x) \cos \left( \frac{n\pi x}{\ell} \right) \, dx = \begin{cases} 
-\frac{4\ell^2}{n^2 \pi^2} & \text{if } n \text{ even} \\
0 & \text{if } n \text{ odd}
\end{cases} \]
One may thus write the Fourier cosine series for the given function as
\[ \frac{\ell^2}{6} - \sum_{k=0}^{\infty} \frac{4\ell^2}{n^2 \pi^2} \cos \left( \frac{(2k+1)\pi x}{\ell} \right). \]
We similarly conclude that this series converges to the given function on the interval \([0, \ell]\), and its even and \(2\ell\)-periodic extension on all of \( \mathbb{R} \).

(iii) We conclude the decay rate of Fourier sine coefficients is approximately \( \sim n^{-3} \) while the decay rate of Fourier cosine series is approximately \( \sim n^{-2} \), so the sine coefficients decay faster. Observe from above that the even extension of \( \phi \) is only continuous on \( \mathbb{R} \) but not differentiable (at \( x = 0 \)), and its odd extension is continuously differentiable on all \( \mathbb{R} \). Therefore Fourier cosine series \( \phi_c \) is only continuous on \( \mathbb{R} \) but not differentiable, while the Fourier sine series \( \phi_s \) is \( C^1 \). This leads us to suspect (correctly, in this case) that a higher degree of differentiability usually indicates a more rapid decay in the Fourier coefficients. And the fact that you don’t need to know is, the biggest possible \( k \) so that the Fourier series coefficients \( a_n \) satisfies \( n^k a_n \) is bounded is equal to the biggest order that the function \( f \) is (weakly) differentiable.
Problem 8.

(i) Observe that \( \phi(\ell) = \ell \), and hence the \( L^2 \)-limit of the Fourier sine series is discontinuous at the points \( \{ n\ell : n \in \mathbb{Z} \} \). Hence the convergence of the Fourier sine series cannot be uniform, since the uniform limit of continuous functions is continuous.

(ii) We know that the \( L^2 \)-limit of the Fourier sine series \( \phi_s \) for \( \phi \) is an odd \( 2\ell \)-periodic function which is smooth away from the points \( \{ n\ell : n \in \mathbb{Z} \} \). However, one easily verifies that \( \phi_s \) is differentiable at these points since \( \phi(0) = \phi(\ell) = 0 \). Hence \( \phi_s \in C^1(\mathbb{R}) \) and therefore Theorem 0.2 of the handout implies that the Fourier sine series converges uniformly.

Problem 9.

(i) Clearly, we have
\[
|C_n| = \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} e^{-inx} \phi(x) dx \right| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |e^{-inx} \phi(x)| dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\phi(x)| dx,
\]
and this last expression is simply \( \frac{1}{2\pi} \| \phi \|_{L^1} \).

(ii) First observe that it follows directly from the definition of \( C_n \) that
\[
-2\pi C_n = \int_{-\pi}^{\pi} \phi(x)e^{-inx}e^{i\pi} dx = \int_{-\pi}^{\pi} \phi(x)e^{-in(x-\pi/n)} dx
\]
\[
= \int_{-\pi}^{\pi} \phi_{\text{per}}(x + \pi/n)e^{-inx} dx.
\]
Subtracting this expression from the one given for \( C_n \), we find that
\[
4\pi C_n = \int_{-\pi}^{\pi} [\phi_{\text{per}}(x + \pi/n) - \phi(x)] e^{-inx} dx \Rightarrow 4\pi |C_n| \leq \| \phi^{(\pi/n)} \|_{L^1}.
\]
Here, of course, \( \phi^{(\pi/n)}(x) = \phi_{\text{per}}(x + \pi/n) \). A standard argument using the triangle inequality on \( L^1 \) and the density of continuous functions in \( L^1([-\pi, \pi]) \) shows that \( \| \phi^{(\pi/n)} \|_{L^1} \to 0 \) as \( n \to \infty \), and hence we conclude \( |C_n| \to 0 \) as \( n \to \infty \), so we are done.

Problem 10. Consider the set of all sequences of continuous functions which are Cauchy with respect to the usual \( L^1 \) norm (defined with respect to the Riemann integral), and let \( L^1([0,1]) \) denote the set of equivalence of all Cauchy sequences, where we say two sequences \( \{a_n\}, \{b_n\} \) are equivalent if \( \{a_1, b_1, a_2, b_2, \ldots\} \) is
Cauchy. We define $L^2([0,1])$ analogously. Thus if $\{\varphi_n\}$ represents an element of $L^2$, given any $\epsilon > 0$ there is some $N$ such that

$$\int_0^1 |\varphi_m - \varphi_n|^2 < \epsilon$$

for every pair $m,n \geq N$. But for such pairs, an easy application of Cauchy-Schwarz gives

$$\int_0^1 |\varphi_m - \varphi_n| \leq \left(\int_0^1 |\varphi_m - \varphi_n|^2\right)^{1/2} \left(\int_0^1 1^2\right)^{1/2} < \sqrt{\epsilon}.$$ 

We thus conclude that $\{\varphi_n\}$ is Cauchy with respect to the $L^1$-norm we have defined, and so $L^2 \subset L^1$. This implies that the identity map $\iota$ on $C([0,1])$ extends to a map $L^2 \to L^1$ (where, as usual, a continuous function $\psi$ is represented in either completion by the sequence $(\psi) := \{\psi, \psi, \psi, \ldots\}$). Moreover, by a similar argument one concludes that $\iota$ is continuous as a map $L^2 \to L^1$, for it again follows from the Cauchy-Schwarz inequality and the compactness of the interval $[0,1]$ that

$$\int_0^1 |\varphi| \leq \left(\int_0^1 |\varphi|^2\right)^{1/2},$$

and hence

$$\|\{\varphi_n\}\|_{L^1} = \lim_{n \to \infty} \int_0^1 |\varphi_n| \leq \lim_{n \to \infty} \left(\int_0^1 |\varphi_n|^2\right)^{1/2} = \|\{\varphi_n\}\|_{L^2},$$

where we are abusing notation slightly to denote by “$\{\varphi_n\}$” the equivalence class containing the sequence $\{\varphi_n\}$ in either completion. Hence the extension $\iota : L^2 \to L^1$ is bounded and thus continuous.

To show that this is the unique continuous extension of $C([0,1])$ with these properties, suppose that $\iota'$ is another. Indeed we have

$$\|\iota\{\varphi_n\} - \iota'\{\varphi_n\}\|_{L^1} \leq \|\iota\{\varphi_n\} - \iota(\varphi) + \iota'(\varphi) - \iota'(\varphi_n)\|_{L^1}$$

$$\leq \|\iota\{\varphi_n\} - \iota(\varphi)\|_{L^1} + \|\iota'(\varphi) - \iota'(\varphi_n)\|_{L^1} \to 0$$

as $k \to \infty$ by the continuity of $\iota, \iota'$. We thus conclude that $\iota = \iota'$.

To show that $\iota : L^2 \to L^1$ is injective, we argue as in the hint. Supposing $\iota$ were not injective, we could find some $\{\varphi_n\}$ such that $\|\{\varphi_n\}\|_{L^2} \neq 0$ while $\|\{\varphi_n\}\|_{L^1} = 0$. One easily checks that this implies there is some continuous function $\varphi_m \in \{\varphi_n\}$ such that, for every $n \geq N$, we have $\langle \varphi_n, \varphi_m \rangle_{L^2} \geq \delta > 0$. Now define a map $F : L^2 \to C$ by $\{\psi_n\} \mapsto \lim_{n \to \infty} \langle \psi_n, \varphi_m \rangle$. It’s easy to check that this map is well-defined. Moreover, since $\varphi_m$ is continuous on $[0,1]$, there is some $M > 0$ such that $|\varphi_m| \leq M$, and hence

$$|F(\{\psi_n\})| = \left|\lim_{n \to \infty} \int_0^1 \psi_n \varphi_m\right| \leq \lim_{n \to \infty} \int_0^1 |\psi_n| |\varphi_n| \leq M \|\{\psi_n\}\|_{L^1}.$$ 

This proves that $F$ extends to a continuous map $L^1 \to C$. Notice that if $\|\{\varphi_n\}\|_{L^1} = 0$, then $F(\{\varphi_n\}) = 0$, but this contradicts the definition of $\varphi_m$. This contradiction proves that $\iota : L^2 \to L^1$ is injective.