Math 172  
Problem Set 3 Solutions

1.4a Denote by \( \hat{C}_n \) the set obtained after the \( n \)th step in the construction of \( \hat{C} \). By construction, \( m(\hat{C}_n) = 1 - \sum_{k=1}^{n} 2^{k-1} \ell_k \). Each \( \hat{C}_n \) is obviously measurable, and hence so is \( \hat{C} = \cap_n \hat{C}_n \). It thus follows directly from monotonicity that \( m(\hat{C}) = 1 - \sum_{k=1}^{\infty} 2^{k-1} \ell_k \).

1.4b Each \( \hat{C}_n \) is the union of \( 2^n \) intervals of equal length. Since a subinterval is removed from the center of each interval to obtain \( \hat{C}_{n+1} \) from \( \hat{C}_n \), for any \( x \in \hat{C} \) there is some point \( x_n \) belonging to one of the intervals removed in the \( n \)th step such that \( |x - x_n| < 2^{-n} \). Since \( x_n \notin \hat{C} \) for all \( n \), we conclude that the sequence \( \{x_n\} \) satisfies the given property.

1.4c The proof of this is essentially identical to the proof of exercise 1.1. Part b above implies that any open interval containing a point in \( \hat{C} \) also contains points outside of \( \hat{C} \), so this set contains no open intervals. On the other hand since each of the intervals comprising \( \hat{C}_n \) are of length less than \( 2^{-n} \), we conclude that any neighborhood of a point \( x \in \hat{C} \) contains the endpoints of one such interval. Since these endpoints lie in \( \hat{C} \) by construction, \( x \) cannot be isolated and therefore \( \hat{C} \) is perfect.

1.4d A point belongs to \( \hat{C} \) if an only if it belongs to every \( \hat{C}_n \). Hence for each \( x \in \hat{C} \), we can produce a sequence \( \{a_i\} \) where \( a_i \in \{0, 1\} \) as follows. If \( x \) lies in the first interval of \( \hat{C}_1 \), set \( a_1 = 0 \) and set \( a_1 = 1 \) otherwise. The interval of \( \hat{C}_k \) in which \( x \) lies is broken into two intervals belonging to the collection of \( 2^{k+1} \) disjoint intervals comprising \( \hat{C}_{k+1} \). If \( x \) lies in the first of these, set \( a_{k+1} = 0 \) and set \( a_{k+1} = 1 \) otherwise. Continuing in this way produces an infinite sequence of 1’s and 0’s for each \( x \), and the map sending points of \( \hat{C} \) to such sequences is easily seen to be a bijection. Since the latter set is uncountable, so is \( \hat{C} \).

17. Note first that since \( f_n \) is measurable, so is \( f_n/c \) for any \( c \neq 0 \). It follows that for any such \( c \), the set \( \{x : f_n(x)/c > 1/n\} \) is measurable. For a fixed \( n \), the set of points \( x \) where \( f_n(x)/c > 1/n \) holds for all \( c \) is precisely the set of all \( x \) satisfying \( |f_n(x)| = \infty \). This set has measure zero by assumption, so Corollary 3.3 in the text tells us that \( m(\{x : f_n(x)/c > 1/n\}) \to 0 \) as \( c \to \infty \). This means that for each \( n \), there is a \( c_n \) such that \( m(\{x : f_n(x)/c_n > 1/n\}) < 2^{-n} \). Having chosen
this sequence \( \{ c_n \} \), the set of points \( \{ x : f_n(x)/c_n \neq 0 \} \) \( \subset \{ x : f_n(x)/c_n > 1/n \) for infinitely many \( n \} = \cap_n \cup_{k \geq n} \{ x : f_k(x)/c_k > 1/k \}. \) However, this set has measure zero by the Borel-Cantelli lemma since the sum \( \Sigma_n 2^{-n} \) converges. This proves that \( f_n(x)/c_n \to 0 \) for almost every \( x \).

32a. By construction, \( \mathcal{N} \cap (\mathcal{N} + q) = \emptyset \) for any rational \( q \). This means that if \( E \) is any measurable subset, we also have \( E \cap (E + q) = \emptyset \). Now let \( \{ q_i \} \) be any infinite sequence of rational numbers in \([0,1]\). Since each translate of \( E \) is measurable, we see that

\[
m_{\infty} \left( \bigcup_{i=1}^{\infty} E + q_i \right) = \sum_{i=1}^{\infty} m(E + q_i) = \sum_{i=1}^{\infty} m(E) = \infty
\]

if \( m(E) > 0 \). This is a contradiction because the union of all these translates lives in \([-1,2] \), so we must have \( m(E) = 0 \).

32b. Since \( m_*(G) > 0 \), there must be some interval \( I = [n,n+1] \) for which \( m_*(G \cap I) > 0 \). Now let \( \{ q_k \} \) be an enumeration of the rationals in \([0,1]\). By the construction of the nonmeasurable set \( \mathcal{N} \) above, we know that \( I \subset \cup_i (\mathcal{N} + q_i) \), and so

\[
0 < m_*(G \cap I) \leq m_* \left( \bigcup_{i=1}^{\infty} G \cap (N + q_i) \right) \leq \sum_{i=1}^{\infty} m_*(G \cap (N + q_i)).
\]

There is thus some \( \epsilon \) such that \( m_*(G \cap (N + q_i)) > 0 \). But then \( G \cap (N + q_i) - q_i \) is a subset of \( \mathcal{N} \) with positive outer measure. This set must be nonmeasurable by part a, and therefore so is its translate by \( q_i \), which is a nonmeasurable subset of \( G \).

33. Assume that \( U \) is as in the hint. The complement of \( U \) in \( \mathbb{R} \) is measurable, and hence so is its complement in \([0,1]\), namely \( V = U^c \cap [0,1] \). By Theorem 3.2 in the text, \( 1 = m_*([0,1]) = m_*(U \cup V) = m_*(U) + m_*(V) \). Hence \( m_*(V) > \epsilon \), but \( V \) is a subset of \( \mathcal{N} \) by definition, so the measurability of \( V \) contradicts exercise 32a above. We thus conclude that \( m_*(\mathcal{N}) = 1 \). Now if \( m_*(\mathcal{N}) = 0 \), it would be measurable by Property 2 of measurable sets, so we necessarily have \( m_*(\mathcal{N}) > 0 \). Hence \( m_*(\mathcal{N}) + m_*(\mathcal{N}^c) = 1 = m_*(\mathcal{N} \cup \mathcal{N}^c) \).

1.34 Let \( C_{\xi} \) and \( C_{\xi'} \) be two Cantor sets as in exercise 1.3. If \( C_{\xi}^k, C_{\xi'}^k \) have their usual meaning, we may write \( C_{\xi}^k \) as the union of disjoint closed intervals

\[
C_{\xi}^k = \bigcup_{n=1}^{2^k} I_n^k
\]
where the intervals are ordered in the standard way (namely if \( i < j \), \( x \in \hat{I}_i^k \), \( y \in \hat{I}_j^k \), then \( x < y \)). Similarly, \((C_k^\xi)^c\) is a disjoint union of open intervals

\[
(C_k^\xi)^c = \bigcup_{j=1}^{2^{j-1}} I_n^j.
\]

Each \( I_n^j \) has length \( \xi^j \), and each \( \hat{I}_i^k \) has length \( 2^{-j}(1 - \xi)^j \). We define

\[
C_k^\xi = \bigcup_{n=1}^{2^k} \hat{J}_n^k \quad \text{and} \quad (C_k^\xi)^c = \bigcup_{j=1}^{2^{j-1}} J_n^j
\]

analogously.

Now we define \( F : C_k^\xi \to C_k^\xi \) by prescribing \( F|_{I_n^k} \) to be the increasing linear function such that \( F(I_n^k) = J_n^k \). Notice that \( F \) is defined almost everywhere on \([0, 1]\), and it is continuous, increasing, and injective on its domain of definition. Next, we extend \( F \) to all of \([0, 1]\) as follows. We know from exercise 1.4 that for each \( x \in C_k^\xi \), we can always find either a strictly increasing sequence such that \( x_n \uparrow x \) or a strictly decreasing sequence such that \( x_n \downarrow x \) and \( x_n \in C_k^\xi \) for each \( n \). For any such sequence, \( \lim_{n \to \infty} F(x_n) \) exists since \( F \) is increasing and \([0, 1]\) is bounded. We define \( F(x) \) to be this limit. If we can show that this definition gives a well defined function on \([0, 1]\), we immediately have that \( F \) is injective because it is strictly increasing. Moreover, one easily checks that \( F(I_n^k) = \hat{J}_n^k \), and so the injectivity of \( F \) implies

\[
F(C_k^\xi) = F \left( \bigcap_{k=1}^{\infty} C_k^\xi \right) = \bigcap_{k=1}^{\infty} F(C_k^\xi) = \bigcap_{k=1}^{\infty} C_k^\xi = C_k^\xi.
\]

To prove that \( F \) is well defined, assume for contradiction that \( x_n \to x \) and \( x'_n \to x \) while \( \lim_{n \to \infty} F(x_n) < \lim_{n \to \infty} F(x'_n) \). There are two cases to consider:

**Case 1:** One of the limits lies in some \( J_n^k \). Suppose \( \lim_{n \to \infty} F(x_n) \) is this limit. This implies that \( \text{dist}(F(x_n), C_k^\xi) \geq \epsilon > 0 \) for sufficiently large \( n \), and hence for such an \( n \) all \( x_n \) lie in \( I_n^k \) and \( \text{dist}(x_n, C_k^\xi) \) is bounded away from zero. This, of course, is a contradiction since \( x_n \to x \in C_k^\xi \).

**Case 2:** Both limits lie in \( C_k^\xi \). In this case we may find some \( k \) such that \( \lim_{n \to \infty} F(x_n) \in \hat{J}_n^k := [a_1, b_1] \) and \( \lim_{n \to \infty} F(x'_n) \in \hat{J}_{n+1}^k := [a_2, b_2] \). Hence for any \( \epsilon > 0 \) there is some sufficiently large \( n \) such that \( F(x_n) < b_1 + \epsilon \) and \( F(x'_n) > a_2 - \epsilon \), and therefore

\[
F(x'_n) - F(x_n) > a_2 - b_1 - 2\epsilon = (\xi')^k - 2\epsilon \Rightarrow x'_n - x_n > \left( \frac{\xi}{\xi'} \right)^k (\xi'^k - 2\epsilon).
\]

Since \( \epsilon \) was arbitrary, we can choose it so that this quantity is positive, which is a contradiction because \( x'_n - x_n \to 0 \). This proves that \( F \) is well defined.
Next we will show that $F$ is continuous. Note that the continuity of $F$ together with the fact that $F(0) = 0$ and $F(1) = 1$ implies that $F$ is surjective by the intermediate value theorem. We again need to check three cases to ensure continuity of $F$ at $x$:

**Case 1:** $x \in C^b_\ell$. In this case continuity is obvious because $F$ is linear in a neighborhood of $x$.

**Case 2:** $x \in C_\ell$ and $x$ is not an endpoint of some interval $\hat{I}_{im}$ for any $k, m$. First note that this case covers points such as the real number with the ternary expansion .0202020202... in the standard Cantor set. In this case we can find a sequence $\{(k_i, m_i)\}$ such that $x \in \text{int}(\hat{I}_{im})$ for all $i$ (here int() denotes the interior). Hence given any $\epsilon > 0$, choose $k_i$ such that $2^{-k_i}(1 - \xi^i)^{k_i} < \epsilon$. Then there exists some $\delta > 0$ such that $|x - x'| < \delta$ implies $x' \in \text{int}(\hat{I}_{im})$. Hence $F(x), F(x') \in \hat{I}_{im}$, and therefore $|F(x) - F(x')| < 2^{-k_i}(1 - \xi^i)^{k_i} < \epsilon$.

**Case 3:** $x \in C_\ell$ and $x$ is an endpoint of some $\hat{I}_{im}$. Note that this implies that $x$ is an endpoint of $\hat{I}_{im}$ for an increasing sequence of $k$'s. Suppose $x$ is a left endpoint. For any $\epsilon > 0$, if $x' < x$ we can use the definition of $F$ on the open interval adjacent to $\hat{I}_{im}$ explicitly compute that if $x - x' < (\xi/\xi^i)^k \epsilon$, then $F(x) - F(x') < \epsilon$. On the other hand, if $x' > x$, we again have that $x' \in \text{int}(\hat{I}_{im})$ whenever $x' - x$ is sufficiently small, so we can simply apply the same argument as in the previous case to that $F$ is continuous from the right. An analogous argument applies if $x$ is a right endpoint.

Now that we have shown $F : [0, 1] \to [0, 1]$ to be continuous, by the observations above we conclude that $F$ has the desired properties.

1.35 Let $\Phi$ be as in the hint, and first note that $\Phi(N)$ is measurable since it is the subset of a set with outer measure zero. Hence if $f = \chi_{\Phi(N)}$, $f \circ \Phi$ cannot be measurable. If it were, we would have $(f \circ \Phi)^{-1}([1, \infty)) = \Phi^{-1} \circ f^{-1}([1, \infty)) = \Phi^{-1}(\Phi(N)) = N$ as a measurable set, a contradiction.

Next consider the set $\Phi(N)$. We have already observed that this set is measurable, and we claim that it is not Borel. Indeed, since $\Phi$ is a continuous function, the inverse image of any open set is open. Moreover, countable unions and intersections and complements are also preserved by $\Phi^{-1}$, and so the inverse image of any set in the $\sigma$-algebra generated by the open sets under these operations (i.e. Borel sets) lands in the same $\sigma$-algebra (i.e. Borel algebra) under $\Phi^{-1}$. However, all Borel sets are measurable, and therefore $N = \Phi^{-1}(\Phi(N))$ is not Borel. We thus conclude that $\Phi(N)$ is not Borel.

2.3 Let $I = [a, b]$. The hint is equivalent to the fact that $I$ contains some point $(2n + 1)\pi$ where $n \in \mathbb{Z}$. Using the additivity property of Proposition 1.6 (of
Chapter 2 in the text), we have
\[ \int_a^b f = \int_a^{(2n+1)\pi} f + \int_{(2n+1)\pi}^b f. \] (0.1)

Now use the translational invariance of the Lebesgue integral to conclude
\[ \int_a^{(2n+1)\pi} f = \int_{a-2\pi}^{\pi} f \quad \text{and} \quad \int_{(2n+1)\pi}^b f = \int_{-\pi}^{b-(2n+2)\pi} f. \] (0.2)

However, we know that \((b - (2n + 2)\pi) - (a - 2n\pi) = b - a - 2\pi = 0\), and therefore we use (0.1), (0.2), and the additivity property again to conclude
\[ \int_a^b f = \int_{a-2\pi}^{\pi} f + \int_{-\pi}^{b-(2n+2)\pi} f = \int_{-\pi}^{\pi} f. \]

2.6a We follow the hint and define a function
\[ g = \sum_{n=2}^{\infty} \frac{n}{n^3} \chi_{[n,n+1/n^3)} \Rightarrow \int_{\mathbb{R}} g = \sum_{n=2}^{\infty} n \cdot m \left( \left[ n, n + \frac{1}{n^3} \right) \right) = \sum_{n=2}^{\infty} \frac{1}{n^2} < \infty. \]
g is thus integrable, and clearly \(\limsup_{x \to \infty} g = \infty\), but \(g\) is not continuous. We define a new function \(\hat{f}\) on the interval \([n - 1/2^n, n]\) to be linear interpolation between the points \((n - 1/2^n, 0)\) and \((n, n)\), and similarly define \(\hat{f}\) on the interval \([n + 1/n^3, n + 1/n^3 + 1/2^n]\) to be linear interpolation between the points \((n + 1/n^3, n)\) and \((n + 1/n^3 + 1/2^n, 0)\). If we set \(\hat{f} = g\) everywhere else, it is easy to see that \(\hat{f}\) is continous and \(\limsup_{x \to \infty} \hat{f} = \infty\). Moreover, it is integrable because
\[ \int_{\mathbb{R}} |\hat{f}| = \int_{\mathbb{R}} \hat{f} = \int_{\mathbb{R}} g + \sum_{n=2}^{\infty} \frac{1}{2^n} = \int_{\mathbb{R}} g + \frac{1}{2} < \infty. \]

Finally, in order to define a function \(f\) with this property which is everywhere positive, we can simply take any positive integrable function \(h\) (take \(h(x) = e^{-x^2}\), for example) and set \(f = \hat{f} + h\). Clearly \(\limsup_{x \to \infty} f = \infty\), while
\[ \|f\|_{L^1} \leq \|\hat{f}\|_{L^1} + \|h\|_{L^1} < \infty. \]

\(f\) is thus integrable so we are done.

2.6b Suppose \(\limsup_{x \to \infty} f \geq c > 0\), and we had a uniform continuity condition such as for every \(x\) and every \(\epsilon > 0\), there exists a uniform \(\delta > 0\) such that \(|x - x'| < \delta \Rightarrow |f(x) - f(x')| < \epsilon\). Now the \(\limsup\) condition implies that there exists a sequence of points \(\{x_n\}\) such that \(x_{j+1} - x_j > 1\) (so that \(x_n \to \infty\)) where \(f(x_n) \geq c/2\). Choosing \(\epsilon < c/4\), the uniform continuity condition then implies
that there is some $\delta > 0$ such that $f > c/4$ on each interval $(x_n - \delta, x_n + \delta)$. But this implies
\[ f > \frac{\sum_{n=1}^{\infty} c}{4} \chi_{(x_n - \delta, x_n + \delta)} \Rightarrow \int_{\mathbb{R}} |f| > \frac{\sum_{n=1}^{\infty} 2\delta c}{4} = \infty. \]
Hence $f$ is not integrable. The contradiction implies that if $f$ is integrable, we must have $\limsup_{x \to \infty} f = 0$. Applying the same argument to $-f(x)$ and $f(-x)$ implies $\lim_{|x| \to 0} f = 0$.

2.9 Since $f \geq 0$, we have by construction that $f \geq \alpha \chi_{E_\alpha}$. Now since $f$ is integrable and thus measurable, the set $E_\alpha$ is measurable, so we conclude that
\[ \infty > \int_{\mathbb{R}^d} |f| = \int_{\mathbb{R}^d} f \geq \int_{\mathbb{R}^d} \alpha \cdot \chi_{E_\alpha} = \alpha \cdot m(E_\alpha) \Rightarrow m(E_\alpha) \leq \frac{1}{\alpha} \int_{\mathbb{R}^d} f. \]

2.10 We first verify that the first sum is finite if and only if the second is. One direction is trivial. Namely, $E_{2k}$ is the disjoint union $F_k \cup F_{k+1} \cup \cdots$, so in particular
\[ F_k \subset E_{2k} \Rightarrow m(F_k) \leq m(E_{2k}) \Rightarrow \sum_{k=-\infty}^{\infty} 2^k m(F_k) \leq \sum_{k=-\infty}^{\infty} 2^k m(E_{2k}). \]

For the other direction, observe that since $m(E_{2k}) = m(F_k) + m(F_{k+1}) + \cdots$, we have
\[ \sum_{n=-N}^{N} 2^n m(E_{2n}) = \sum_{n=-N}^{N} 2^n \left( \sum_{k=-\infty}^{\infty} m(F_k) \right) = \sum_{k=-N}^{\infty} \left( \sum_{j=-\infty}^{\inf(N,k)} 2^j \right) m(F_k). \]
So letting $N \to \infty$, we find that
\[ \sum_{n=-\infty}^{\infty} m(E_{2n}) = \sum_{k=-\infty}^{\infty} \left( \sum_{j=-\infty}^{\inf(N,k)} 2^j \right) m(F_k) = 2 \sum_{k=-\infty}^{\infty} 2^{k+1} m(F_k) = 2 \sum_{k=-\infty}^{\infty} 2^k m(F_k), \]
proving the other direction.

To show that $f \geq 0$ being integrable implies that the first (and thus second) sum is finite, first observe that each $F_k$ is measurable by the measurability of $f$. Now by construction, $f > 2^k \chi_{F_k}$ for all $k$. Since the $F_k$’s are disjoint, we thus have
\[ \int_{\mathbb{R}^d} f > \int_{\mathbb{R}^d} \sum_{k=-N}^{N} 2^k \chi_{F_k} = \sum_{k=-N}^{N} 2^k m(F_k). \]
Taking $N \to \infty$ shows that the first sum is bounded above by the integral of $f$, and so the first sum and second sums are finite. On the other hand, it’s clear that
\[ f \leq \sum_{k=-\infty}^{\infty} 2^{k+1} \chi_{F_k} \Rightarrow \int_{\mathbb{R}^d} f \leq \sum_{k=-\infty}^{\infty} 2^{k+1} m(F_k), \]
but we just showed that this last quantity is equal to the value of the second sum. So if the second sum is finite, $f$ must be integrable.

Next we show that the function

$$f(x) = \begin{cases} \frac{|x|^a}{d} & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

is integrable if and only if $a < d$. We need only show that the second sum in the first part of the problem is finite if and only if this is true. Now for $k \leq 0$, $E_{2k} = B_1$, the closed unit ball, so in this case $m(E_{2k}) = \omega_d$. For $k > 1$, we have $E_{2k} = \{x : |x|^{-a} > 2^k\} = \{x : |x| < 2^{-k/a}\}$, which is the open unit ball of radius $2^{-k/a}$. Hence

$$\sum_{k=-\infty}^{\infty} 2^k m(E_k) = \sum_{k=0}^{0} 2^k \omega_d + \sum_{k=1}^{\infty} \omega_d \frac{1}{2} = \omega_d \left(\frac{2}{1 - 2^{-k/a}}\right).$$

This last sum is a geometric series, which converges if and only if $1 - 2^{-k/a} < 0$ if and only if $a < d$.

The argument for integrability of the second function

$$g(x) = \begin{cases} \frac{|x|^{-b}}{d} & \text{if } |x| > 1 \\ 0 & \text{otherwise} \end{cases}$$

is similar. $g$ is obviously not integrable if $b \leq 0$, so assume $b > 0$. In this case $E_{2k} = \emptyset$ for $k \geq 0$. Moreover, for $k < 0$, we have $E_{2k} = \{x : |x|^{-b} > 2^k\} = B_2 \cdot 2^{-k/b} - B_1$. We thus have $m(E_{2k}) = \omega_d (2^{-kd/b} - 1)$, so that

$$\sum_{k=-\infty}^{\infty} 2^k m(E_{2k}) = \sum_{k=0}^{0} 2^k \omega_d (2^{-kd/b} - 1) = \omega_d \left(\frac{2}{1 - 2^{-k/b}}\right) - \omega_d.$$

Again, the final sum is a geometric series which converges if and only if $1 - 2^{-k/b} > 0$ if and only if $a < d$.

2.16 Given a measurable set $E$ and a $d$-tuple $(\delta_1, \ldots, \delta_d)$ with each $\delta_i > 0$, we define the set $E^\delta = \{(\delta_1 x_1, \ldots, \delta_d x_d) : (x_1, \ldots, x_d) \in E\}$. Recall that we showed in exercise 1.7 (in problem set 1) that

$$m(E^\delta) = \delta_1 \cdots \delta_d m(E).$$

A nearly identical proof shows that if we let $\delta_i < 0$ as well, we in fact have

$$m(E^\delta) = |\delta_1| \cdots |\delta_d| m(E).$$

Now first suppose $f > 0$. It follows from Theorem 4.1 in Chapter 1 of the text that we may find an increasing sequence of nonnegative simple functions $s_i$ which converges to $f$ pointwise. It then follows from the dominated convergence theorem that $f$ is integrable.
theorem that, in fact, \( s_i \to f \) in \( L^1 \). Of course the pointwise convergence implies that \( s_i(\delta x) \to f(\delta x) = f^\delta(x) \) for all \( x \) as well, so we again conclude from the dominated convergence theorem that \( s_i^\delta \to f^\delta \) in \( L^1 \). Next observe that

\[
s_i = \sum_{j=1}^{N_i} a_{ij} \chi_{E_{ij}} \Rightarrow s_i^\delta = \sum_{j=1}^{N_i} a_{ij} \delta \chi_{E_{ij}} = \sum_{j=1}^{N_i} a_{ij} \chi_{E_{ij}^{1/\delta}},
\]

where we are using the notation \( E_{ij}^{1/\delta} = \{(\delta_1^{-1}x_1, \ldots, \delta_d^{-1}x_d) : (x_1, \ldots, x_d) \in E \} \). We have

\[
\int_{\mathbb{R}^d} \chi_{E_{ij}^{1/\delta}} = m(E_{ij}^{1/\delta}) = |\delta_1|^{-1} \cdots |\delta_d|^{-1} m(E),
\]

and therefore

\[
\int_{\mathbb{R}^d} s_i^\delta = |\delta_1|^{-1} \cdots |\delta_d|^{-1} \int_{\mathbb{R}^d} s_i.
\]

The \( L^1 \)-convergence of \( s_i^\delta \to f^\delta \) thus implies that

\[
\int_{\mathbb{R}^d} f^\delta = |\delta_1|^{-1} \cdots |\delta_d|^{-1} \int_{\mathbb{R}^d} f.
\]

Of course the general case follows by applying this argument to both \( f_+ \) and \( f_- \) separately.