Problem 1. (i) Let $K_P$ be the one-dimensional Poisson kernel:
\[
K_P(r, \omega) = \frac{1}{2\pi}\frac{1 - r^2}{1 - 2r \cos \omega + r^2}.
\]
Show that for $f \in L^1([-\pi, \pi])$, $f * K_P \rightarrow f$ in $L^1$ as $r \uparrow 1$.

(ii) Show that $2\pi$-periodic $C^\infty$ functions are dense in $L^1([-\pi, \pi]^d)$.

Problem 2. Consider the wave equation on a ring of length $2\ell$. We let $x$ be the arclength variable along the ring, $x \in [-\ell, \ell]$. We would like to understand wave propagation along the ring, so consider the wave equation with periodic boundary conditions:
\[
u_{tt} = c^2 u_{xx}, \quad u(-\ell, t) = u(\ell, t), \quad u_x(-\ell, t) = u_x(\ell, t).
\]
(i) Show that if the initial conditions are $u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x)$, with $\phi, \psi$ $2\ell$-periodic $C^{k+1}$, $k \geq 2$, functions then there is a solution $u \in C^k(\mathbb{R} \times \mathbb{R}_t)$, $2\ell$-periodic in $x$ of the wave equation satisfying the initial and boundary conditions.

(ii) Show that if the initial conditions are $u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x)$, with $\phi, \psi$ $2\ell$-periodic $C^3$, then there is at most one $C^3(\mathbb{R} \times \mathbb{R}_t)$, $2\ell$-periodic in $x$, solution of the wave equation.

(iii) Find the solution with initial condition
\[
u(x, 0) = 0, \quad u_t(x, 0) = \cos(2\pi x/\ell) - \sin(\pi x/\ell), \quad x \in [-\ell, \ell].
\]

Problem 3. Consider the (real-valued) heat equation on a rod of length $\ell$ with insulated ends and $k > 0$:
\[
u_t = ku_{xx}, \quad u_x(0, t) = 0 = u_x(\ell, t).
\]

(i) Show that if the initial condition is $u(x, 0) = \phi(x), \phi \in L^2([0, \ell])$, then there is a solution $u \in C^\infty([0, \ell] \times (0, \infty))$ such that the family of function $U(t)(x) = u(x, t)$, depending on the parameter $t > 0$, tends to $\phi$ in $L^2$ as $t \to 0$.

(ii) Show that if $\phi$ is $C^1$, then the convergence of $U(t)$ to $\phi$ is uniform.

(iii) Show that if $u \in C^2([0, \ell] \times (0, \infty))$, satisfying the PDE and the boundary condition, then $E(t) = \int_0^\ell u(x, t)^2$ is monotone decreasing. (Hint: what is $E'(t)$?)

(iv) Show that if $\phi \in L^2([0, \ell])$, then there is a unique solution $u \in C^2([0, \ell] \times (0, \infty))$ of the heat equation satisfying the boundary conditions and with $U(t) \to \phi$ in $L^2$ as $t \to 0$.


Problem 6. Suppose that $f \in L^1(\mathbb{R}^n)$. Throughout this problem, $a \in \mathbb{R}^n$.

(i) Let $f_a(x) = f(x - a)$. Show that $(\mathcal{F} f_a)(\xi) = e^{-ia \cdot \xi}(\mathcal{F} f)(\xi)$.

(ii) Let $g_a(x) = e^{ia \cdot x} f(x)$. Show that $(\mathcal{F} g_a)(\xi) = (\mathcal{F} f)(\xi - a)$.

(iii) Show that $(\mathcal{F}^{-1} f_a)(x) = \frac{1}{(2\pi)^n} e^{ia \cdot x} f(x)$.

(iv) Show that $(\mathcal{F}^{-1} g_a)(x) = \frac{1}{(2\pi)^n} f(x + a)$.

Problem 7. Use part (i) of Problem 6 to show that $(\mathcal{F} (\partial_x f))(\xi) = i\xi_x (\mathcal{F} f)(\xi)$ if $f$ is $C^1$ and $|x|^N f, |x|^N \partial_x f$ are bounded for some $N > n$. 

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