Problem 1. Do Stein-Shakarchi, vol. 3, Ch. 4, Exercise 5.


Problem 4. Do Stein-Shakarchi, vol. 3, Ch. 4, Exercise 8. Here a unitary map is an invertible isometry $U : H_1 \to H_2$ between Hilbert spaces, so $U$, $U^{-1}$ are linear maps preserving norms (and thus inner products by the parallelogram identity).

Problem 5. Let $\phi(x) = |x|$ on $[-\pi, \pi]$. Let $f(x) = a_0 + a_1 \cos x + a_2 \cos(2x) + b_1 \sin x + b_2 \sin(2x)$.

With what choice of the coefficients $a_j$ and $b_j$ is the $L^2([-\pi, \pi])$ error $\|f - \phi\| = \|f - \phi\|_{L^2}$ minimal?


Problem 7. Let $\phi(x) = x(\ell - x)$ on $[0, \ell]$.

(i) Find the Fourier sine series of $\phi$, and state what it converges to for $x \in \mathbb{R}$.

(ii) Find the Fourier cosine series of $\phi$, and state what it converges to for $x \in \mathbb{R}$.

(iii) Compare the decay rates of the coefficients of the two series as $n \to \infty$.

Why do the coefficients decay faster in one of the cases?

Problem 8. For both of the following functions prove or disprove whether the Fourier sine series converges uniformly:

(i) $\phi(x) = x$ on $[0, \ell]$,

(ii) $\phi(x) = x(\ell - x)^2$ on $[0, \ell]$.

You do not need to compute the respective Fourier series.

Problem 9. Suppose $\phi \in L^1([-\pi, \pi])$, and define the Fourier coefficients of $\phi$ by extending the $L^2$-definition:

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} \phi(x) \, dx.$$

(i) Show that the sequence $\{C_n\}_{n=-\infty}^{\infty}$ is bounded, and sup $|C_n| \leq \frac{1}{2\pi} \|\phi\|_{L^1}$ in fact.

(ii) Show that $\lim_{|n| \to \infty} C_n = 0$.

Problem 10. For the purposes of this problem, define $L^1([0, 1])$ and $L^2([0, 1])$ as the completions of $C([0, 1])$ in the $L^1$, resp. $L^2$-norm (using the Riemann integral).

Without using the measure theoretic concepts we developed, show that $L^2([0, 1]) \subset L^1([0, 1])$, i.e. the identity map $C([0, 1]) \to C([0, 1])$ has a unique continuous extension to a map $\iota : L^2([0, 1]) \to L^1([0, 1])$ (show both existence and uniqueness), and this map is injective.

(Hint: For the injectivity, for $f \in L^2([0, 1])$ show that there is $\phi \in C([0, 1])$ such that $\langle f, \phi \rangle \neq 0$, and then show that the map $L^2 \ni g \mapsto \langle g, \phi \rangle \in \mathbb{C}$ extends continuously to a map $L^1 \to \mathbb{C}$.)