1. Hand-in exercises

Problem 1.1 (40.1). Prove Theorem 40.4 (i), (iii), (iv), (v), and (vi).

(i). First, we claim that if \(a, b \in \mathbb{R}\) then \(|a| - |b| \leq |a - b|\). To see this notice that the triangle inequality gives that \(|a| \leq |b| + |a - b|\) and so \(|a| - |b| \leq |a - b|\). Similarly we can get that \(|b| - |a| \leq |a - b|\). Then it follows that \(|a| - |b| \leq |a - b|\).

For \(\varepsilon > 0\), fix \(x\) and find \(\delta > 0\) such that \(d(x, y) < \delta\) then \(|f(x) - f(y)| < \varepsilon\). Then it follows from the paragraph above that \(||f(x)| - |f(y)|| \leq |f(x) - f(y)| < \varepsilon\). Hence \(|f|\) is continuous.

(iii). If \(c = 0\) then this is clearly true since in this case \(cf(x) = 0\) for every \(x \in M\). In the case that \(c \neq 0\), fix \(\varepsilon > 0\) and \(x \in M\) and find \(\delta > 0\) such that \(d(x, y) < \delta\) implies that \(|f(x) - f(y)| < \varepsilon/|c|\). Hence it follows that if \(d(x, y) < \delta\) then \(|cf(x) - cf(y)| = |c||f(x) - f(y)| < \varepsilon\).

(iv). Since \(g\) is continuous, then it follows from (iii) that \(-g\) is continuous. Then it follows from (ii) that \(f + (-g) = f - g\) is continuous.

(v). Fix \(x \in M\) and suppose that \(x_n \in M\) is a sequence converging to \(x\). Then, by continuity, we have that \(\lim f(x_n) = f(x)\) and \(\lim g(x_n) = g(x)\). Hence, by the properties of limits, it follows that \(\lim (f \cdot g)(x_n) = \lim f(x_n)g(x_n) = f(x)g(x)\). Thus, by Theorem 40.2, \(f \cdot g\) is continuous.

(vi). Notice that if we show that \(1/g\) is continuous, then it follows from (v) that \(f/g = f \cdot (1/g)\) is continuous. Hence suppose that \(x \in M \setminus \{y : g(y) = 0\}\) and suppose that \(x_n \in M \setminus \{y : g(y) = 0\}\) is a sequence converging to \(x\). Then since \(g\) is continuous we have that \(\lim g(x_n) = g(x)\) and so, by the properties of limits, we have that \(\lim 1/g(x_n) = 1/g(x)\). Hence \(1/g\) is continuous on \(M \setminus \{y : g(y) = 0\}\).

Problem 1.2 (40.7). Suppose that \(f\) is a function from \((M_1, d_1)\) to \((M_2, d_2)\). Prove that the following are equivalent:

(a) \(f\) is continuous on \(a\)
(b) If \(U\) is an open subset of \(M_2\) which contains \(f(a)\), there exists an open subset \(V\) of \(M_1\) which contains \(a\) such that \(V \subset f^{-1}(U)\)

Proof. First we prove that (a) implies (b). Fix \(U\) as above and fix \(\varepsilon > 0\) such that \(B_\varepsilon(f(a)) \subset U\). Then find \(\delta > 0\) such that \(d_1(a, y) < \delta\) implies that \(d_2(f(a), f(y)) < \varepsilon\), which exists by continuity of \(f\). Let \(V = B_\delta(a)\). Notice that this gives that \(V \subset f^{-1}(B_\varepsilon(f(a))) \subset f^{-1}(U)\).

Now we prove that (b) implies (a). Fix \(\varepsilon > 0\) and let \(U = B_\varepsilon(f(a))\). Then find \(V\) as in the statement of the theorem. Since \(V\) is open, then there exists \(\delta > 0\) such that \(B_\delta(a) \subset V\). Hence if \(d_1(a, y) < \delta\), then \(y \in V\) and so \(f(y) \in U\). By definition of \(U\), \(d_2(f(y), f(a)) < \varepsilon\). Hence if \(d_1(a, y) < \delta\) then \(d_2(f(a), f(y)) < \varepsilon\), i.e. \(f\) is continuous at \(a\).

Problem 1.3 (40.11). Let \(\{a_n\} \in l^2\). Define \(f : l^2 \to \mathbb{R}\) by \(f(\{b_n\}) = \sum_n a_nb_n\). Prove that \(f\) is continuous.

Proof. Notice first that by the discussion in the preceding chapters \(l^2\) is a normed vector space over \(\mathbb{R}\) (with norm \(\|\{c_n\}\|_{l^2} = \sqrt{\sum c_n^2}\)). Hence it follows by the work on your writing assignment that we need only prove that \(f\) is bounded (as it is clearly linear) to show that \(f\) is continuous.

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To this end we simply apply Theorem 36.6, the Cauchy-Schwarz inequality to get that for any \( \{b_n\} \in \ell^2 \):

\[
|f(\{b_n\})| \leq \|\{a_n\}\|_{\ell^2} \|\{b_n\}\|_{\ell^2}
\]

Hence \( f \) is bounded by the constant \( \|\{a_n\}\|_{\ell^2} \), and so is continuous. \( \square \)

**Problem 1.4** (40.17). (a) Prove that the following are equivalent

(i) \( d \) and \( d' \) are equivalent metrics

(ii) The collection of closed subsets of \( (M, d) \) is identical to the collection of closed subsets of \( (M, d') \)

(iii) The sequence \( \{x_n\} \) converges in \( (M, d) \) if and only if \( \{x_n\} \) converges in \( (M, d') \)

(b) Prove that the metrics \( d, d', \) and \( d'' \) of Exercise 35.7 are equivalent.

(c) Prove that the metric \( d' \) of Exercise 37.9 is equivalent to the usual metric on \( \mathbb{R}^n \).

(a). We Will prove that (i) \( \Rightarrow \) (iii) \( \Rightarrow \) (ii) \( \Rightarrow \) (i).

(i) \( \Rightarrow \) (iii): Suppose, by contradiction, that in \( d \), \( \lim x_n = x \) but that \( x_n \) does not converge to \( x \) in \( d' \). In this case we can find a subsequence \( x_{n(k)} \) and \( \delta > 0 \) such that \( d'(x_{n(k)}, x) \geq \delta \) for every \( k \). Let \( B'_k(x) \) be a ball in \( (M, d') \). Notice that for every \( k \), \( x_{n(k)} \notin B'_k(x) \). Since \( B'_k(x) \) is open in \( (M, d') \) then it must be open in \( (M, d) \) and so there exists \( r > 0 \) such that \( B_r(x) \subseteq B'_k(x) \) where \( B_r(x) \) is a ball in \( (M, d) \). Since \( x_{n(k)} \) converges to \( x \) in \( (M, d) \) then there exists \( k \) such that \( x_{n(k)} \in B_r(x) \subseteq B'_k(x) \). This is a contradiction. Hence we must have that \( \lim x_n = x \) in \( (M, d') \).

Similarly it follows that if \( x_n \) converges to \( x \) in \( (M, d') \) then it converges in \( (M, d) \) as well.

(iii) \( \Rightarrow \) (ii): Let \( K \) be closed in \( (M, d) \). Fix any limit point \( x \in M \) and then there exists \( x_n \in K \) such that \( \lim x_n = x \) in \( (M, d') \). Then by (iii) we have that \( \lim x_n = x \) in \( (M, d) \). Hence \( x \) is a limit point of \( K \) under \( d \). Since \( K \) is closed under \( d \) then it follows that \( x \in K \). Hence, \( K \) contains all its limit points under \( d' \). Hence \( K \) is closed in \( (M, d') \). Similarly it follows that if \( K \) is closed in \( (M, d') \) then it is closed in \( (M, d) \) as well.

(ii) \( \Rightarrow \) (i): If \( U \) is open in \( (M, d) \) then \( U' \) is closed in \( (M, d) \) and so \( U' \) is closed in \( (M, d') \). Hence \( U \) is open in \( (M, d') \). Similarly we argue that if \( U \) is open in \( (M, d') \) then \( U \) is open in \( (M, d) \).

(b). This follows from (iii) above and Exercise 37.10.

(c). This follows from (iii) above and Exercise 37.9(b).

**Problem 1.5** (41.1). Deduce Theorem 41.2(ii) from Theorem 41.2(i) using Theorem 29.5.

Proof. First assume that \( Y \) is closed. Then \( X \setminus Y \) is open and so \( X \setminus Y = X \cap U \) where \( U \) is open in \( M \). Then \( Y = X \setminus (X \setminus Y) = X \setminus U = X \cap U' \). Notice that \( U' \) is closed so we are finished.

Now assume that \( Y = X \cap K \) where \( K \) is closed. Then, as above, \( Y' = X \cap K' \). Since \( K' \) is open then \( Y' \) is open (by Theorem 41.2(ii)). Hence \( Y \) must be closed.

**Problem 1.6** (41.5). Let \( M \) be a metric space and let \( X \) be a subset of \( M \) with the relative metric. If \( Y \) is a subset of \( X \), let \( Y^{-(X)} \) denote the closure of \( Y \) in \( X \). Prove that \( Y \cap X = Y^{-(X)} \). State and prove the corresponding result for \( Y^o \).

**Proof.**

\[
Y^{-(X)} = \bigcap_{K \subset X \text{ closed}, K \supseteq Y} K
\]

\[
= \bigcap_{K \subset M \text{ closed, } K \supseteq Y} X \cap K \quad \text{by Exercise 41.1}
\]

\[
= X \cap \overline{\bigcap_{K \subset M \text{ closed, } K \supseteq Y} K}
\]

\[
= X \cap \overline{Y}
\]
Similarly, we define $Y^o(X)$ to be the interior of $Y$ in the relative topology of $X$. We claim that $Y^o(X) = X \cap (X' \cup Y)^o$.

$$
Y^o(X) = \bigcup_{U \subset X \text{ open, } Y \cap U} U
= \bigcup_{U \subset M \text{ open, } X' \cup Y \cap U} X \cap U
= X \cap \left( \bigcup_{U \subset M \text{ open, } X' \cup Y} U \right)
= X \cap (X' \cup Y)^o
$$

since $Y \subset X$. 

\[ \square \]

**Problem 1.7** (42.2). Let $X$ be a compact subset of metric space $M$. Prove that $X$ is closed.

**Proof.** As suggested in the hint, we proceed by showing that $x' \in X$ is open. Let $y \in X'$ and for each $x \in X$ we can find $U_x, V_x$ which are open subsets of $M$ with the property that $V_x \cap U_x = \emptyset$ and $x \in U_x$, $y \in V_x$ (take, for instance, $U_x = B_{d/2}(x)$ and $V_x = B_{d/2}(y)$ where $d = d(x, y)$). Since \{ $U_x \cap X : x \in X$ \} is an open cover of $X$ we can find $x_1, \ldots, x_n$ such that $U_{x_1}, \ldots, U_{x_n}$ covers $X$. Let $V = V_{x_1} \cap \cdots \cap V_{x_n}$. First notice that $y \in V$ since $y \in V_x$ for every $x$. Also notice that $V_x \cap U_x = \emptyset$ implies that $V \cap U_x = \emptyset$ for every $i$. Hence $V \cap X \subset V \cap \left( U_{x_1} \cup \cdots \cup U_{x_n} \right) = \emptyset$. Hence $y$ is contained in an open subset of $X'$. Since this is true for every $y \in X'$ then it follows that $X'$ is open and thus, that $X$ is closed. 

\[ \square \]

**Problem 1.8** (42.3). Let $X_1, \ldots, X_n$ be a finite collection of compact subsets of a metric space $M$. Prove that $X = X_1 \cup \cdots \cup X_n$ is compact. Show that this does not generalize to infinite unions.

**Proof.** Take an open cover $\mathcal{U}$ of $X$. Then for each $i$ we can find a finite subcover $U_{i,1}, \ldots, U_{i,k_i}$ of $X_i$ since $\mathcal{U}_i = \{ U \cap X_i : U \in \mathcal{U} \}$ is an open cover of $X_i$. Then $U_{1,1}, \ldots, U_{k_1,1}, U_{1,2}, \ldots, U_{k_2,2}, \ldots, U_{1,n}, \ldots, U_{k_n,n}$ is a finite open cover of $X$. Hence $X$ is compact.

Notice that for every $n$, $[n, n + 1]$ is compact. However, $\bigcup_{n=1}^{\infty} [n, n + 1] = [1, \infty)$ is not compact since, for example, \{(a, b) : a, b \in \mathbb{R}, a - b = 1 \} is an open cover of $[1, \infty)$ which has no finite subcover. 

\[ \square \]

**Problem 1.9** (42.4). Let $\mathcal{C}$ be a collection of compact subsets of a metric space. Prove that $\bigcap \mathcal{C}$ is compact.

**Proof.** Let $C = \bigcap \mathcal{C}$ and let $\mathcal{U}$ be an open cover of $C$. Fix $K \in \mathcal{C}$ and notice that $C \subset K$. Let $\mathcal{U}' = \mathcal{U} \cup \{ C' \}$. Since $C$ is the intersection of closed sets then $C$ is closed and hence $C'$ is open. Thus $\mathcal{U}'$ is an open cover of $K$. Hence there exists $U_1, \ldots, U_n$ such that $K \subset U_1 \cup \cdots \cup U_n$. If $U_i \neq C'$ for every $i$ then this is a finite subset of $\mathcal{U}$ which covers $C$ (since it is a cover of $K$ and $C \subset K$ and we are finished. If, without loss of generality, $U_1 = C'$ then we claim that $U_2, \ldots, U_n$ is a cover of $C$. Indeed, if $x \in C$ then $x \in U_i$ for some $i$. Since $x \in C$ then $x \notin C'$ and so $x \notin U_1$. Hence $x \in U_i$ for some $2 \leq i \leq n$. Hence $C \subset U_2 \cup \cdots \cup U_n$. Hence, $U_2, \ldots, U_n$ is a finite subset of $\mathcal{U}$ which covers $C$. Hence $C$ is compact.

\[ \square \]

**Problem 1.10** (42.6). Let $f$ be a continuous real-valued function on a compact metric space $M$. Suppose that $f(x) > 0$ for all $x \in M$. Prove that there exists $T > 0$ such that $f(x) > T$ for all $x \in M$. 


Proof. Define $g(x) = 1/f(x)$. By Exercise 40.1, $g$ is continuous on $M$. Hence by Theorem 42.6, there exists $M$ such that $0 < g(x) < M$ for every $x$. Hence we get that $0 < 1/M < f(x)$ for every $x$.

Alternatively, one could use that there exists $c \in M$ such that $f(c) \leq f(x)$ for every $x \in M$. Letting $T = f(c)$ finishes the proof. \hfill \square

**Problem 1.11** (42.12). Prove that if $f$ is a contractive mapping on a compact metric space $M$, there exists a unique point $x$ such that $f(x) = x$.

**Proof.** Define $g(x) = d(f(x), x)$. We first claim that $g(x)$ is continuous. To see this, fix $x$ and $\epsilon > 0$ and find $\delta' > 0$ such that $d(x, y) < \delta'$ implies that $d(f(x), f(y)) < \epsilon/2$. Let $\delta = \min\{\delta', \epsilon/2\}$ and if $d(x, y) < \delta$ then $g(x) - g(y) = d(f(x), x) - d(f(y), y) \leq d(f(x), f(y)) + d(f(y), y) + d(y, x) - d(f(y), y) < \epsilon/2 + \epsilon/2$

Similarly we can easily get that

$g(y) - g(x) < \epsilon$

Hence if $d(x, y) < \delta$ then $|g(x) - g(y)| < \epsilon$. This implies that $g$ is continuous.

By Corollary 42.7, there exists $z \in M$ such that if $x \in M$ then $g(z) \leq g(x)$. Now if $g(z) = 0$ then we’re finished because then $z = f(z)$. On the other hand, if $g(z) > 0$ then notice that $g(f(z)) < g(z)$ (since $f$ is a contractive mapping). This is clearly a contradiction. Hence $f(z) = z$ and so $z$ is the desired fixed point.

Now suppose $y \in M$ is another fixed point of $f$. We get a contradiction by noticing that

$d(z, y) < d(f(z), f(y)) = d(z, y)$

Hence, $z$ must be the unique fixed point. \hfill \square

2. Selected other problems

**Problem 2.1** (42.10). Let $\{X_n\}$ be a sequence of compact subsets of a metric space $M$ with $X_1 ⊃ X_2 ⊃ X_3 ⊃ \ldots$. Prove that if $U$ is an open set containing $\cap X_n$, then there exists $X_n \subset U$.

**Proof.** We do a proof by contradiction and claim that if $X_n \not\subseteq U$ for all $n$, then $\cap X_n \setminus U = (\cap X_n) \setminus U \neq \emptyset$, contradicting $\cap X_n \subset U$. Being a closed subset of a compact subset, $X_n \setminus U \subset X_n$ is compact as well. Furthermore $X_n \setminus U$ contains $X_{n+1} \setminus U$.

So it suffices to prove that if $M \supset C_1 \supset C_2 \supset C_3 \supset \ldots$ is a decreasing sequence of non-empty compact subsets, then their intersection is non-empty. To see this, note that if their intersection is empty, then $\{M \setminus C_i\}$ is an open cover of $C_1$. Hence there is a finite subcover and by the nestedness of the $C_i$ we must have $M \setminus C_k \supset C_1$. But this is impossible, as $C_k$ is non-empty and contained in $C_1$. \hfill \square