1. Hand-in exercises

Problem 1.1 (16.11). Prove directly, from definition 10.2, that \( \lim_{n \to \infty} n^{1/n} = 1 \).

Proof. As suggested in the hint let \( \delta_n = n^{1/n} - 1 \). Notice that if \( n > 1 \) then \( n^{1/n} > 1 \). Hence \( \delta_n > 0 \) for every \( n \geq 2 \). Then using some algebra we get that

\[
\delta_n = (\delta_n + 1)^n = \sum_{k=0}^{n} \binom{n}{k} \delta^k n^k \geq \binom{n}{2} \delta^2 = \frac{1}{2} n(n-1) \delta^2
\]

Here we get the inequality since all terms in the binomial sum are positive. Hence we have that for \( n > 2 \), \( \delta_n \leq \frac{2}{n-1} \). Hence

\[
|n^{1/n} - 1| = \delta_n \leq \frac{2}{n-1}
\]

Since this last term tends to 0, then it follows that \( \lim n^{1/n} = 1 \). \( \square \)

Problem 1.2 (25.3). Let \( \{a_n\} \) be a decreasing sequence with limit 0. Let \( L = \sum_{n=1}^{\infty} (-1)^{n+1} a_n \). Show that

\[
|L - s_n| \leq a_{n+1}
\]

where \( s_n \) is the \( n \)th partial sum.

Proof. Define \( L_n = \sum_{k=n+1}^{\infty} (-1)^{k+1} a_k \). If \( n = 2m \) is even then notice that \( L_n = \sum_{k=m}^{\infty} (a_{2k+1} - a_{2k+2}) \). Since \( a_k \) is a decreasing sequence then \( L_n \) is the sum of all non-negative terms. Hence \( L_n \) is non-negative. Similarly we show that \( L_n \) is non-positive if \( n \) is odd.

If \( n \) is even then we have that:

\[
|L - s_n| = L_n = a_{n+1} + L_{n+1} \leq a_{n+1}
\]

If \( n \) is odd then we have that:

\[
|L - s_n| = -L_n = a_{n+1} - L_{n+1} \leq a_{n+1}
\]

Problem 1.3 (26.1). Determine whether each series diverges, converges, or converges absolutely.

(a). This series converges absolutely since \( \sum \frac{1}{n} \) converges and since \( |(-1)^{n+1} n^{1/n}| \leq \frac{1}{n^2} \) for every \( n \) \( \square \)

(b). This series converges but does not converge absolutely. To see that it converges notice that it is an alternating series and that \( \frac{1}{n^{1+1/n}} \) is decreasing, then apply Alternating Series Test. To see that it doesn’t converge uniformly notice that \( |(-1)^{n+1} n^{1/n}| \geq \frac{1}{2n} \) for every \( n \) and notice that \( \sum \frac{1}{2n} \) does not converge. \( \square \)

(c). This converges absolutely. Notice that

\[
\sqrt{n+1} - \sqrt{n} = (\sqrt{n+1} - \sqrt{n}) \left( \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \right) = \frac{1}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{2\sqrt{n}}
\]

Hence \( |(-1)^{n+1} (\sqrt{n+1} - \sqrt{n})| \leq n^{-3/2}/2 \). Since \( \sum n^{-3/2} \) converges then we are finished. \( \square \)

(d). This converges absolutely since \( \frac{|(-1)^{n+1} n^{1/n}|}{2^{n}(n+1)} \leq 2^{-n} \) and since \( \sum 2^{-n} \) converges. \( \square \)

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(e). This converges absolutely. First we notice that \( \lim_{x \to \infty} x^4 2^{-x} = 0 \). To see this, simply use L’Hospital’s rule. Hence we have that \( \lim_{n \to \infty} n^4 2^n = 0 \). Let \( C = \sup_{n \in \mathbb{P}} n^4 2^{-n} \). Since this is a convergent sequence then it is bounded and so \( C \) is well-defined. Hence \( n^4 2^{-n} \leq C \) for every \( n \).

Now we return to our original problem and notice that:

\[
\left| \frac{(-1)^{n+1} n^2}{2^n} \right| = \frac{n^4}{n^2 2^n} \leq C
\]

Since \( \sum n^{-2} \) is convergent then our origin series must be absolutely convergent.

\( \square \)

(f). This converges absolutely. To see this notice that \( (2n-1)! \geq (2n-3)^2 \) for \( n \geq 2 \), and hence, \( \frac{(-1)^{n+1}}{(2^n-1)!} \leq (2n-3)^{-2} \) for all \( n \geq 2 \). Since \( \sum (2n-3)^{-2} \) converges, then our series must converge absolutely.

\( \square \)

**Problem 1.4** (26.4). Prove that if \( \sum a_n \) converges absolutely, then \( \sum a_n^2 \) converges.

*Proof.* Fix \( N \) such that \( n \geq N \) implies that \( |a_n| \leq 1 \). Then if \( n \geq N \) we have that \( a_n^2 \leq |a_n| \).

Hence we have that \( \sum_{n=N}^{\infty} a_n^2 \) must converge since \( \sum |a_n| \) converges. Hence \( \sum_{n=1}^{\infty} a_n^2 \) converges as well, since we are only adding finitely many terms to the series.

\( \square \)

**Problem 1.5** (26.10). Prove that if \( x \in [0,1] \) then there exists a sequence \( a_1, a_2, \ldots \) such that \( a_n \in \{0,1,\ldots,9\} \) for every \( n \) and such that \( \sum a_n = x \). (Notice this is just saying that every element in \([0,1]\) has a decimal expansion)

*Proof.* We define the sequence inductively. Choose \( a_1 \) to be the largest integer less than 10 such that \( a_1/10 \leq x \). Since \( x > 0 \) then \( a_1 \geq 0 \) and since \( x < 1 \) then we have that \( 0 \leq x - \frac{a_1}{10} \leq 1/10 \). Now assume that we have chosen \( a_1, \ldots, a_n \) to be integers of the set \( \{0,\ldots,9\} \) such that \( 0 \leq x - s_n \leq 10^{-n} \), where \( s_n = \frac{a_1}{10} + \cdots + \frac{a_n}{10^n} \). Choose \( a_{n+1} \) to be the largest integer less than 10 such that \( a_{n+1}/10^{n+1} \leq x - s_n \). Just as before we get that \( a_{n+1} \geq 0 \) and, since \( x - s_n = 10^{-n} \), then it must hold that \( 0 \leq x - s_n - a_{n+1} \leq 10^{-n} \). Hence \( 0 \leq x - s_{n+1} \leq 10^{-n-1} \).

Notice that our choice of \( a_n \) ensures that \( s_n \) converges to \( x \).

\( \square \)

**Problem 1.6** (27.1). Find the radius of convergence, \( R \), of each of the power series. Discuss the convergence of the power series at the points \( |x - t| = R \).

(a). This has infinite radius of convergence, i.e. \( R = \infty \). To see this we will show that \( (n!)^{-1/n} \) converges to \( 0 \). To see this, fix \( \epsilon > 0 \) and find \( N \) such that \( m \geq N \) implies that \( 1/m < \epsilon^3 \). Now if \( n > 2N \) we let \( m \) be such that \( 2m = n \) or \( 2m = n - 1 \). Then \( m \geq N \) and so

\[
0 \leq \left( \frac{1}{n!} \right)^\frac{1}{n} \leq \left( \frac{1}{m \cdot (m-1) \cdots n} \right)^\frac{1}{n} \leq \left( \frac{1}{m^m} \right)^\frac{1}{n} = m^{-m/n} \leq n^{-1/2n} \leq m^{-1/2+1/2n} \leq m^{-1/3} < \epsilon
\]

since \( m \geq \frac{n-1}{2} \) 

Now since \( a_n = 0 \) if \( n \) is even and \( a_n = 1/n! \) if \( n \) is odd, then the above shows that \( \lim \sup |a_n|^{1/n} = 0 \). Hence \( R = \infty \).

\( \square \)

(b). We know from problem 16.11 that \( \lim n^{1/n} = 1 \). Hence we have that

\[
\lim \sup |a_n|^{1/n} = \lim \sup \left( \frac{n}{2^n} \right)^{1/n} = \frac{1}{2} \lim \sup n^{1/n} = \frac{1}{2}
\]
Hence $R = 2$. Now when $|x - 1| = 2$ we have that either $x = -1$ or $x = 3$. In the first case we have
\[
\sum n(\frac{\frac{1}{n}}{-2})^n = 0.
\]
and in the second case, where $x = 3$, we easily see that our sum becomes $\sum n$. This clearly does not converge as well. 

(c). Here $R = 0$. To see this, notice that our proof earlier that $\lim (n!)^{-1/n} = 0$ then $\lim (n!)^{1/n} = \infty$, and so we compute:

\[
\lim \sup |a_n|^{1/n} = \lim \sup \frac{(n!)^{1/n}}{(2n!)^{1/n}} = \lim \sup \frac{1}{2} = \infty.
\]

\[
\Box
\]

**Problem 1.7 (27.2).** Suppose that $\sum a_n(x - t)^n$ has radius of convergence $R$. If $p$ is an integer, show that $\sum n^p a_n(x - t)^n$ has radius of convergence $R$.

**Proof.** First notice that since $\lim n^{1/n} = 1$ then it follows that $\lim (n^p)^{1/n} = 1$ as well. Then we compute:

\[
\lim \sup |n^p a_n|^{1/n} = \lim \sup \frac{|n^p a_n|^{1/n}}{(2n^p)^{1/n}} = \lim \sup \frac{|a_n|^{1/n}}{2} = \infty.
\]

Here the second equality comes from Theorem 20.8. Hence $\sum n^p a_n(x - t)^n$ must have the same radius of convergence as $\sum a_n(x - t)^n$. 

\[
\Box
\]

**Problem 1.8 (29.1).** Let $\sum a_n(x - t)^n$ and $\sum b_n(x - t)^n$ have radii of convergence $R_1$ and $R_2$ respectively. If $|x - t| < R$ where $R = \min\{R_1, R_2\}$ then $\sum (a_n + b_n)(x - t)^n$ converges absolutely and is equal to $\sum a_n(x - t)^n + \sum b_n(x - t)^n$.

**Proof.** Fix $x$ such that $|x - t| < R$. Notice that $|(a_n + b_n)(x - t)^n| \leq |a_n||x - t|^n + |b_n||x - t|^n$.

Let $A_n = \sum_{n=0}^{\infty} |a_n||x - t|^n$ and let $B_n = \sum_{n=0}^{\infty} |b_n||x - t|^n$. Let $C_n = \sum_{n=0}^{\infty} |a_n + b_n||x - t|^n$. Notice that $C_n$ is a monotonically increasing sequence so it suffices to show that it is bounded to show that it converges. Notice also that $A_n$ and $B_n$ are monotonically increasing sequences. By

our assumptions there exist $A, B < 2$ such that $\lim A_n = A$ and $\lim B_n = B$. Then we get

\[
C_n \leq A_n + B_n \leq A + B
\]

The first inequality comes from what we noticed above and the second comes from the fact that $A_n$ and $B_n$ are monotonically increasing sequences. Hence $C_n$ is bounded, i.e. $\sum (a_n + b_n)(x - t)^n$ converges absolutely.

We will redefine $A_n, B_n$ and $C_n$ now to be $A_N = \sum_{n=0}^{N} a_n(x - t)^n$, $B_N = \sum_{n=0}^{N} b_n(x - t)^n$, and $C_N = \sum_{n=0}^{N} (a_n + b_n)(x - t)^n$. Then notice that, for every $n$, $A_n + B_n = C_n$.

Hence we get

\[
\sum (a_n + b_n)(x - t)^n = \lim C_n = \lim (A_n + B_n) = \lim A_n + \lim B_n = \sum a_n(x - t)^n + \sum b_n(x - t)^n
\]

\[
\Box
\]

**Problem 1.9 (29.3).** Prove that $\sum \frac{x^n}{n!}$ converges absolutely for every $n$ and that $\left(\sum \frac{x^n}{n!}\right) \left(\sum \frac{y^n}{n!}\right) = \sum \frac{(x+y)^n}{n!}$ for every $x, y$.

**Proof.** The first fact follows from what we proved above, namely that $\lim (n!)^{1/n} = 0$ and hence the series has infinite radius of convergence. For the second fact, fix $x, y \in \mathbb{R}$ and examine

\[
\sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) x^k y^{n-k}
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} x^k y^{n-k}
\]

\[
= \sum_{n=0}^{\infty} \frac{x^k y^{n-k}}{k! (n-k)!}
\]

\[
\Box
\]
Notice that this is simply the Cauchy product of $\sum x^n/n!$ and $\sum y^n/n!$ and so by Theorem 29.9 we get that
\[
\left(\sum x^n/n!\right) \left(\sum y^n/n!\right) = \sum (x+y)^n/n!
\]

\[\blacksquare\]

**Problem 1.10 (29.7).** Let $x$ be any real number and let $\sum a_n$ be a series which converges conditionally but not absolutely. Prove that there is a permutation of $\sum a_n$ which converges to $x$.

**Proof.** Let’s first make some reductions. We may assume that $a_n \neq 0$ for all $n$, since if $f : \mathbb{P} \to \mathbb{P}$ gives the subsequence of $a_n$ which contains exactly the non-zero terms and if we can construct a permutation/rearrangement $g : \mathbb{P} \to \mathbb{P}$ such that $\sum a_{g(f(n))} = x$, then we can get a permutation of the original sequence converging to $x$ as follows. There are two cases. First, if there are infinitely many $n$ such that $a_n = 0$, then let $h : \mathbb{P} \to \mathbb{P}$ be the subsequence of all $a_n = 0$. Then define $k : \mathbb{P} \to \mathbb{P}$ as $k(2n) = g(f(n))$ and $k(2n-1) = h(n)$. It is easy to check that $\sum a_{h(n)} = x$. Otherwise, if there are only $N$ elements of the original sequence which are zero, then let $h : \{1, \ldots, N\} \to \mathbb{P}$ be an injective map such that $a_{h(n)} = 0$ for every $n$. Then define $k$ as $k(n) = h(n)$ if $n \leq N$ and let $k(n) = g(f(n-N))$ if $n > N$. It is again easy to check that $\sum a_{k(n)} = x$.

Also, notice that $a_n$ must have infinitely many positive elements and infinitely many negative elements, since if it has only finitely many positive (respectively negative) elements in the sequence, then absolute and conditional convergence are the same. Thus, let $f : \mathbb{P} \to \mathbb{P}$ give the subsequence of all positive elements and let $g : \mathbb{P} \to \mathbb{P}$ give the subsequence of all negative elements. Since any sequence of positive elements which converges to zero can be rearranged to be monotonically decreasing, and similarly any sequence of negative elements which converges to zero can be rearranged to be monotonically increasing, we may assume that $f, g$ are monotonic functions.

Finally, we assume that $x \geq 0$, but the proof is exactly analogous for negative $x$.

Now we are ready to begin the actual proof. We will define a bijection $k : \mathbb{P} \to \mathbb{P}$ inductively as follows. The idea here is we are going to sum up positive elements until we get just above $x$, then we will add some negative elements to correct for this until we get just below $x$, and then we will continue doing this, first adding positive numbers to get just above $x$ then adding negative numbers until we get just below $x$. The monotonicity of $f$ and $g$ will make it easy to check for convergence.

Let $N_1$ be the first positive integer such that $\sum_{n=1}^{N_1} a_{f(n)} > x$. Notice that $|x - \sum_{n=1}^{N_1} a_{f(n)}| \leq a_{f(N_1)}$. Define $k(n) = f(n)$ for $1 \leq n \leq N_1$. Now find the smallest positive integer $M_1$ such that $\sum_{n=1}^{N_1} a_{f(n)} + \sum_{n=M_1}^{M_1} a_{g(n)} < x$. Define $k(n+1) = g(n)$ for $1 \leq n \leq M_1$, and notice that $|x - \sum_{n=1}^{N_1+1} a_{k(n)}| \leq \max\{|a_{g(M_1)}|, |a_{f(N_1)}|\}$ for every $1 \leq l \leq M_1$.

Now assume that we’ve defined $k$ up to $N_m + M_m$ and assume that
\[
|x - \sum_{n=1}^{l+M_m} a_{k(n)}| \leq \max\{|a_{g(M_m)}|, |a_{f(N_m)}|\}
\]
for all $M_{m-1} < l \leq M_m$, that
\[
|x - \sum_{n=1}^{l+M_{m-1}} a_{k(n)}| \leq \max\{|a_{g(M_{m-1})}|, |a_{f(N_{m})}|\}
\]
for all $N_{m-1} < l \leq N_m$ and that $\sum_{n=1}^{N_m+M_m} a_{k(n)} < x$. Now find the smallest positive integer $N_{m+1} > N_m$ such that $\sum_{n=1}^{N_{m+1}} a_{f(n)} + \sum_{n=M_m}^{M_{m+1}} a_{g(n)} > x$. Define $k(n+M_m) = f(n)$ for $N_m < n \leq N_{m+1}$. Again we notice that for every $l$ such that $N_m < l \leq N_{m+1}$ we have that
\[
|x - \sum_{n=1}^{l+M_{m+1}} a_{k(n)}| \leq \max\{|a_{f(N_{m+1})}|, |a_{g(M_m)}|\}
\]
Now find the smallest positive integer $M_{m+1} > M_m$ such that $\sum_{n=1}^{N_{m+1}} a_{f(n)} + \sum_{n=M_{m+1}}^{M_{m+1}} a_{g(n)} < x$. Then define $k(n+N_{m+1}) = g(n)$ for all $M_m < n \leq M_{m+1}$. Notice that as before we have that for
all \( M_m < l \leq M_{m+1} \),
\[
|x - \sum_{n=1}^{N_{m+1}+l} a_k(n)| \leq \max\{|a_g(M_{m+1})|, a_f(N_{m+1})\}
\]

This gives us a one-to-one, onto function \( k : \mathbb{P} \to \mathbb{P} \) and since \( \lim a_f(n) = \lim a_g(n) = 0 \) and since we have the inequalities given above, then it follows easily that \( \sum a_k(n) = x \), as desired. \( \square \)

**Problem 1.11** (29.14). Let \( \sum a_n x^n \) be a power series with radius of convergence \( R > 0 \) and suppose that \( a_0 \neq 0 \). Prove that there exists \( \sum b_n x^n \) with radius of convergence \( R^* > 0 \) such that if \( x < \min\{R, R^*\} \), then
\[
\sum b_n x^n = \frac{1}{\sum a_n x^n}
\]

**Proof.** We assume that \( a_0 = 1 \) since if not, we get let \( c_n = a_n/a_0 \) and if we find \( b_n \) such that \((\sum c_n x^n)(\sum b_n x^n) = 1 \) then \( \sum (b_n/a_0)x^n \) is the series we want.

We wish to find a power series such that
\[
\sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} b_k a_{n-k} = \left(\sum b_n x^n\right) \left(\sum a_n x^n\right) = 1
\]

Since we wish this to be the power series expansion for 1 we want
\[
\sum_{k=0}^{n} b_k a_{n-k} = 0
\]
for every \( n > 0 \). Define \( b_0 = 1 \). Then define
\[
b_n = -\sum_{k=0}^{n-1} b_k a_{n-k}
\]

With this definition, it is easy to check that \( b_n \) satisfies the relation we wanted above. Hence we need only show that \( \sum b_n x^n \) converges with positive radius of convergence. Here are two alternative approaches to this:

1. Define a relation \( \sum a_n x^n \prec \sum b_n x^n \) if for every \( n \), \( |a_n| \prec b_n \). Then notice that if \( f \prec g \prec \phi \) and \( h \prec k \), then \( fh \prec gk \), \( f + h \prec g + k \), \( f \prec \phi \) and that the radius of convergence of \( f \) is larger than that of \( g \). Hence, in our case, it suffices to show that \( \sum b_n x^n \prec \sum c_n x^n \) for some power series \( \sum c_n x^n \) with a positive radius of convergence.

Define \( f(x) = \sum a_n x^n \), write \( h(x) = 1 - f(x) \), and write \( g(x) = \sum b_n x^n \). Then, formally, \( g(x) = \frac{x}{1-\phi(x)} = 1 + h(x) + h(x)^2 + \ldots \). Since \( \lim \sup |a_n|^{1/n} = 1/R \), then let \( A = \sup |a_n|^{1/n} \) and then \( |a_n| \leq A^n \) for every \( n \). Hence \( h \prec \sum A^n x^n = \frac{Ax}{1-Ax} \). Thus we get that
\[
g \prec 1 + \frac{Ax}{1-Ax} + \left(\frac{Ax}{1-Ax}\right)^2 + \ldots
\]
\[
= \frac{1}{1 - \frac{Ax}{1-Ax}}
\]
\[
= (1 - Ax)(1 + 2Ax + (2Ax)^2 + \ldots)
\]
\[
< (1 + Ax)(1 + 2Ax + (2Ax)^2 + \ldots)
\]

Since this last line defines a power series with radius of convergence \( 1/2A \), then \( g \) must have radius of convergence at least \( 1/2A \).

2. Alternatively, let \( A = \sup |a_n|^{1/n} \). Then we have \( |a_n| \leq A^n \). We claim that \( |b_n| \leq 2^{n-1}A^n \) for \( n \geq 1 \), which we’ll prove using induction. Note that \( b_0 = 1 \), and hence \( |b_1| = |b_0a_1| \leq A \).
Thus it is true in the case $n = 1$. Let’s prove that the cases $0 \leq k \leq n$ imply the case $n + 1$:

$$|b_{n+1}| = | - \sum_{k=0}^{n} b_k a_{n+1-k}| \leq \sum_{k=0}^{n} |b_k||a_{n+1-k}|$$

$$\leq \sum_{k=0}^{n} 2^{k-1}A^k A^{n+1-k} = A^{n+1}(1 + \sum_{k=1}^{n} 2^{k-1}) = 2^n A^{n-1}$$

which completes our induction. We conclude that

$$lim sup_{n \to \infty} \frac{|b_n|}{n} \leq lim sup_{n \to \infty} 2^n A = 2A$$

Both approaches tell us that $g$ defines a convergent power series which by Theorem 29.9 satisfies the property that $(\sum_{n=0}^{\infty} b_n x^n)$ $(\sum_{n=0}^{\infty} c_n x^n) = 1$, as desired. \qed

2. Selected additional exercises

**Problem 2.1 (26.5)**. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be absolutely convergent series. Prove that the series $\sum_{n=1}^{\infty} \sqrt{|a_n b_n|}$ converges.

*Proof.* Consider $(\sqrt{|a_n|} - \sqrt{|b_n|})^2 \geq 0$. This is equivalent to $|a_n| + |b_n| \geq 2 \sqrt{|a_n| \sqrt{|b_n|}} = 2 \sqrt{|a_n b_n|}$, which will be the key inequality in this proof.

Since the series $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$ both converge, so does the series $\sum_{n=1}^{\infty} \frac{1}{2}(|a_n| + |b_n|)$ by theorem 23.1. We want to apply the comparison test (theorem 26.3(i)) to $\sqrt{|a_n b_n|}$ and $\frac{1}{2}(|a_n| + |b_n|)$ respectively. We check the hypothesis

$$\sqrt{|a_n b_n|} = \sqrt{|a_n b_n|} \leq \frac{1}{2}(|a_n| + |b_n|) = \frac{1}{2}(|a_n| + |b_n|)$$

where the outer equalities hold because the terms are non-negative and the middle inequality was proven in the beginning of this proof. Using the comparison test we now conclude that since $\sum_{n=1}^{\infty} \frac{1}{2}(|a_n| + |b_n|)$ converges absolutely (convergence in this case is the same as absolute convergence, because all terms are non-negative), so does $\sum_{n=1}^{\infty} \sqrt{|a_n b_n|}$. \qed