Solution Set  
Math 171  
Problem Set 3

**Problem 1 - 19.2**  Prove that \( \{a_n\} \) is a Cauchy sequence if and only if for every \( \epsilon > 0 \) there exists \( N \) such that \( n > N \) implies that \( |a_n - a_N| < \epsilon \).

Assume that \( \{a_n\} \) is a Cauchy sequence. Then fix \( \epsilon > 0 \) and find \( N' \) such that \( n, m > N' \) implies that \( |a_n - a_m| < \epsilon \). Let \( N = N' + 1 \) and if \( n > N \) then \( n, N > N' \) and so we have that \( |a_n - a_N| < \epsilon \).

Now assume that the condition in the problem holds and we'll show that \( \{a_n\} \) is Cauchy. Fix \( \epsilon > 0 \) and find \( N \) such that \( n > N \) implies that \( |a_n - a_N| < \epsilon/2 \). Then if \( n, m > N \) we have that \( |a_n - a_N| < \epsilon/2 \) and that \( |a_m - a_N| < \epsilon/2 \).

\[
|a_n - a_m| \leq |a_n - a_N| + |a_m - a_N| < \epsilon/2 + \epsilon/2 = \epsilon
\]

SIDE NOTE: The first inequality here is called the “triangle inequality.” It is usually stated as: If \( x, y \in \mathbb{R} \) then \( |x + y| \leq |x| + |y| \). It is a very useful inequality to know and proving it amounts to checking all the cases (i.e. if \( x > y > z \), \( x > z > y \), etc.). You should do this on your own.

**Problem 2 - 20.7**  Compute \( \lim \sup a_n, \lim \inf a_n \) and \( \mathcal{L}_a \), where \( a_n \) is an enumeration of the rational numbers in \([0, 1]\).

We first show that \( \mathcal{L}_a = [0, 1] \). Fix \( x \in (0, 1] \). We will define a subsequence converging to \( x \), with the property that \( 0 < x - a_{n_k} < 2^{-k} \), inductively. Find \( n_1 \) such that \( 0 < x - a_{n_1} < 1/2 \). Now assume that we have found \( a_{n_1}, \ldots, a_{n_k} \) such that \( \{a_{n_k}, x - 2^{-(k+1)}\} \). Then notice that \( (x_{n_k}, x) \cap \mathbb{Q} \) is infinite and the set \( a_1, \ldots, a_{n_k} \) is finite. Hence, we can find \( n_{k+1} > n_k \) such that \( a_{n_{k+1}} = \max \{a_{n_k}, x\} \), i.e. \( 0 < x - a_{n_{k+1}} < 2^{-(k+1)} \).

Notice that the sequence \( a_{n_k} \) converges to \( x \) as desired. The case for \( x = 0 \) is almost exactly the same except we choose \( a_{n_k} \) such that \( 0 < a_{n_k} < 2^{-k} \).

Notice that by the definitions of \( \lim \sup \) and \( \lim \inf \), since \( \mathcal{L}_a = [0, 1] \), then \( \lim \sup a_n = 1 \) and \( \lim \inf a_n = 0 \).

**Problem 3 - 20.13**  Let \( \{a_n\} \) and \( \{b_n\} \) be sequences such that the former is convergent and the latter is bounded. Prove that

\[
\lim \sup (a_n + b_n) = \lim \sup a_n + \lim \sup b_n
\]
\[
\lim \inf (a_n + b_n) = \lim \inf a_n + \lim \inf b_n
\]

Let \( L = \lim a_n \) and notice that \( \lim \sup a_n = L \). Let \( M = \lim \sup b_n \). Fix \( \epsilon > 0 \) and choose a convergent subsequence \( b_{n_k} \) with limit \( M' \) such that \( M' + \epsilon \geq M \). Then \( a_{n_k} + b_{n_k} \) is a convergent subsequence of \( a_n + b_n \) which converges to \( L + M' \). Hence \( L + M' \in \mathcal{L}_{a+b} \) and so \( \lim \sup (a_n + b_n) \geq L + M' \geq L + M - \epsilon \).

Since this statement is true for any possible \( \epsilon > 0 \) then it must be true that \( \lim \sup (a_n + b_n) \geq L + M = \lim \sup a_n + \lim \sup b_n \).

Since we already know, by Theorem 20.6, that \( \lim \sup (a_n + b_n) \leq \lim \sup a_n + \lim \sup b_n \), then by this and the above we get that \( \lim \sup (a_n + b_n) = \lim \sup a_n + \lim \sup b_n \).

The proof for the analogous statement for \( \lim \inf \) is done similarly.
Problem 4 - 20.21  Let \( \{a_n\}, \{b_n\}, \{c_n\} \) be bounded sequences such that for every \( n \), \( a_n \leq b_n \leq c_n \) and such that \( \limsup c_n \leq \liminf a_n \). Then \( \lim a_n = \lim b_n = \lim c_n \).

First notice that since \( a_n \leq c_n \) for every \( n \), then

\[
\liminf c_n \leq \limsup c_n \leq \liminf a_n \leq \liminf c_n.
\]

Hence we must have that \( \limsup c_n = \liminf c_n \) and so \( \lim c_n \) exists. Similarly we get that

\[
\limsup a_n \geq \liminf a_n \geq \limsup c_n \geq \limsup a_n
\]

Hence we must have that \( \limsup a_n = \liminf a_n \) and so \( \lim a_n \) exists. Moreover, this gives us that \( \lim a_n = \limsup a_n = \limsup c_n = \lim c_n \). Finally, we apply the Squeeze Theorem to get the desired result.

Problem 5 - 21.4  (a) Let \( \{a_n\} \) be a sequence which is bounded above. Prove that either \( \{a_n\} \) diverges to \( -\infty \), in which case \( \limsup a_n = -\infty \), or that \( \{a_n\} \) has a convergent subsequence, in which case \( \limsup a_n = \text{lub} \ L_a \).

(b) State and prove the analogous result for (a) corresponding to \( \liminf \).

(a) Let \( M \) be such that \( M \geq a_n \) for every \( n \). Now, if \( a_n \) diverges to \( -\infty \) we need to show that \( \limsup a_n = -\infty \). For each positive integer \( k \), there exists \( N_k \) such that \( m \geq N_k \) implies that \( -k > a_m \). Hence letting \( A_m = \text{lub} \ \{a_{m}, a_{m+1}, \ldots \} \) as in the book, we have that \( A_m \leq -k \) for every \( m \geq N_k \). Hence, it must be true that \( \limsup a_n \leq -k \). Since this holds for every \( k \), then we have that \( \limsup a_n = -\infty \).

Now suppose that \( a_n \) does not diverge to \( -\infty \). Thus, by negating the definition of divergence, we get that there exists \( L \) such that for every \( N \) there is \( m \geq N \) such that \( a_m \geq L \). Using this we create a subsequence inductively. For \( N = 1 \) we apply the above definition to get \( n_1 \geq 1 \) such that \( a_{n_1} \geq L \). Now, assume that we’ve picked \( a_{n_1}, \ldots, a_{n_k} \). Then let \( N = n_k + 1 \) and find \( n_{k+1} \geq N \) such that \( a_{n_{k+1}} \geq L \). Then \( a_{n_k} \) is a bounded subsequence since it is bounded above by \( M \) and below by \( L \). Hence it has a convergent subsequence \( a_{m_k} \). Let \( c = \lim a_{m_k} \).

Now we wish to show that, in this case, \( \limsup a_n = \text{lub} \ L_a \). Let \( A_n = \text{lub} \ \{a_n, a_{n+1}, \ldots \} \) as in the book. Notice that \( A_n \geq c \) for every \( n \). Then \( \limsup a_n = \lim A_n \geq c \). Define a subsequence of \( a_n \) as follows. For each \( k \), choose \( a_{n_k} \) such that \( n_k > n_{k-1} \) and such that \( 0 \leq a_{n_{k+1}} - a_{n_k} < 2^{-k} \). Notice then that \( \lim A_k = \lim A_{n_{k+1}} = \lim a_{n_k} \) and so \( \text{lub} \ L_a \geq \lim A_k \) since \( \lim A_k \in L_a \). Now we wish to show the opposite inequality. Fix \( \epsilon > 0 \) and find an element \( b \in L_a \) such that \( \text{lub} \ L_a \leq b + \epsilon \). Let \( a_{n_k} \) be a subsequence which converges to \( b \). Then we have that \( a_n \geq b \) for every \( n \). Hence we have that \( \lim A_n \geq b \geq \text{lub} \ L_a - \epsilon \). Since this is true for any \( \epsilon > 0 \) then it must be true that \( \lim A_n \geq \text{lub} \ L_a \). Putting this together with our other inequality we get that \( \limsup a_n = \lim A_n = \text{lub} \ L_a \).

(b) The statement of the problem is: Let \( \{a_n\} \) be a sequence which is bounded below. Prove that either \( \{a_n\} \) diverges to \( \infty \), in which case \( \liminf a_n = \infty \), or that \( \{a_n\} \) has a convergent subsequence, in which case \( \liminf a_n = \text{glb} \ L_a \).

We skip the proof of this as it will be exactly analogous to the proof for (a).

Problem 6 - 22.3  Suppose that \( |x| < 1 \). Prove that \( \sum_n x^{2^n} \) converges and find its sum.

Since \( |x| < 1 \) then \( |x|^2 < |x| < 1 \). Hence we may apply Theorem 22.4 to get that \( \sum_n y^n \) converges and is equal to \( \frac{1}{1-y} \) where \( y = x^2 \). In other words \( \sum_n x^{2^n} \) converges and is equal to \( \frac{1}{1-x^2} \).

Problem 7 - 22.4  Prove that \( \sum_n \frac{1}{n^{(n+1)}} \) converges and find its sum.
We look at the partial sums here. Notice that for any $N$:
\[
\sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \left( \frac{1}{n} - \frac{1}{n+1} \right) = \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \cdots + \left( \frac{1}{N-1} - \frac{1}{N} \right) + \left( \frac{1}{N} - \frac{1}{N+1} \right) = 1 - \frac{1}{N+1}
\]
Clearly as we take $N$ to $\infty$ we get that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1. In other words $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$. 

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**Problem 8 - 23.1** Show that if $\sum_n a_n = L$ and $\sum_n b_n = M$ then $\sum_n (a_n - b_n) = L - M$ and that $\sum_n ca_n = cL$.

The case of $c = 0$ is obvious so we assume that $c \neq 0$. Fix $\epsilon > 0$ and find $N > 0$ such that $|L - \sum_{n=1}^{m} a_n| < \epsilon/|c|$ for all $m > N$. Then for every $m > N$ we have that
\[
|cL - \sum_{n=1}^{N} ca_n| = |cL - c \sum_{n=1}^{N} a_n| = |c||L - \sum_{n=1}^{N} a_n| < \epsilon
\]
Hence $\sum_n ca_n = cL$.

Now we apply the above, with $c = -1$ and the first part of Theorem 23.1 to conclude that
\[
\sum_n (a_n - b_n) = \sum_n (a_n + (-1)b_n) = \sum_n a_n + \sum_n (-1)b_n = L + (-1)\sum_n b_n = L - M
\]

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**Problem 9 - 23.7** Suppose there are constants $c$ and $d$, where $c \neq d$, such that $\sum_n (a_{2n} + ca_{2n-1})$ and $\sum_n (a_{2n} + da_{2n-1})$ converge. Show that $\sum_n a_n$ converges.

First we claim that if $\sum_n a_{2n}$ converges and if $\sum_n a_{2n-1}$ converges, then $\sum_n a_n = \sum_n (a_{2n} + a_{2n-1})$ converges and we are finished. To see this, find $N_1, N_2$ such that if $n \geq N_1$ and $m \geq N_2$ then $|\sum_{k=1}^{n} a_{2n}| < \epsilon/2$ and $|\sum_{k=1}^{m} a_{2n-1}| < \epsilon/2$. Let $N = 2 \max\{N_1, N_2\}$. If $\ell \geq N$ then we can write $\ell = n + m$ where $m = n$ or $n+1$, depending on whether $\ell$ is even or odd. In either case we get that $n \geq N_1$ and $m \geq N_2$. Hence we have that
\[
|\sum_{k=1}^{\ell} a_n| = |\sum_{k=1}^{n} a_{2n} + \sum_{k=1}^{m} a_{2n-1}| \leq |\sum_{k=1}^{n} a_{2n}| + |\sum_{k=1}^{m} a_{2n-1}| < \epsilon/2 + \epsilon/2 = \epsilon
\]
Hence it must be true that $\sum_n a_n$ converges.

First we show that $\sum_n a_{2n-1}$ converges. Let $x_n = a_{2n} + ca_{2n-1}$ and let $y_n = a_{2n} + da_{2n-1}$. Since $\sum_n x_n$ and $\sum_n y_n$ converge then we know that
\[
(c - d) \sum_n a_{2n-1} = \sum_n (c - d)a_{2n-1} = \sum_n [x_n - y_n] = \sum_n x_n - \sum y_n
\]
This shows that $\sum_n a_{2n-1}$ converges.

Lastly we show that $\sum_n a_{2n}$ converges. Let $x_n$ and $y_n$ be as before. Since $c \neq d$ then we know that either $c \neq 0$ or $d \neq 0$. Assume that $c \neq 0$ but similarly for the other case, where $d \neq 0$. As above we compute
\[
(d/c - 1) \sum_n a_{2n} = \sum_n [(d/c)x_n - y_n] = (d/c) \sum_n x_n - \sum y_n
\]
This shows that $\sum_n a_{2n}$ converges, finishing the proof.
**Problem 10 - 24.3** Show that \( \sum 1/n^2 \leq 2 \).

First notice that the series \( \sum 1/n^2 \) converges by Corollary 24.3. Hence it suffices to show that all the partial sums are bounded by 2. Next notice that for every \( n \geq 2 \) we have that \( 1/n^2 \leq 1/n(n-1) \). Hence we have that

\[
\sum_{n=1}^{N} \frac{1}{n^2} = 1 + \sum_{n=2}^{N} \frac{1}{n^2} \\
\leq 1 + \sum_{n=2}^{N} \frac{1}{n(n-1)} \\
\leq 1 + \sum_{n=1}^{N} \frac{1}{n(n+1)} \\
= 1 + 1 - \frac{1}{N+1} \leq 2
\]

by our work for 22.4

Since this is true for every \( N \) then it must true.

**Problem 11 - 24.9** Prove that if \( a_n \) is a decreasing sequence and \( \sum a_n \) converges then \( \lim na_n = 0 \). Deduce that \( \sum 1/n^s \) diverges if \( 0 \leq s \leq 1 \).

First notice that since \( a_n \) is decreasing and converges to 0, then \( a_n \geq 0 \) for every \( n \).

Fix \( \epsilon > 0 \). Since the sequence of partial sums is convergent then it is Cauchy. Hence we can find \( N \) large enough that \( n, m \geq N \) implies that \( \sum_{k=n}^{m} a_k = |\sum_{k=n}^{m} a_k| < \epsilon \). Then for any \( n \geq N \) we have that

\[
0 \leq na_{2n} \leq a_n + a_{n+1} + \cdots + a_{2n-1} = \sum_{k=n}^{2n-1} a_k < \epsilon
\]

Hence we have that \( \lim na_{2n} = 0 \).

Now fix \( \epsilon > 0 \) and find \( N \) such that \( n \geq N \) implies that \( na_{2n} = |na_{2n}| < \epsilon/3 \). Now for \( n > 2N \) we can find \( m \geq N \) such that \( n/3 \leq m \leq n/2 \). Hence we have that \( |ma_{2m}| < \epsilon/3 \). Since \( a_k \) is a decreasing sequence then we have that \( |ma_n| \leq |ma_{2m}| < \epsilon/3 \). Hence \( |na_n| \leq |3ma_n| < \epsilon \), since \( n \leq 3m \). Thus we have shown that \( \lim na_n = 0 \).

Now suppose that \( \sum 1/n^s \) converges. Then \( 0 = \lim n(1/n^s) = \lim n^{1-s} \). This is clearly false if \( 0 \leq s \leq 1 \). Hence it cannot be true that \( \sum 1/n^s \) converges.