This is a closed book, closed notes, no calculators/computers exam. As usual, \( \mathbb{P} \) denotes the set of positive integers, and \( \mathbb{R}^n \) comes with the Euclidean metric unless otherwise specified.

You may quote any theorem from the textbook or the lecture provided that you are not explicitly asked to prove it (Problem 1(iii) falls in this category), and provided you state the theorem precisely and concisely (make sure to check the hypotheses in writing when you quote a theorem).

There are 5 problems. Solve all of them. Write your solutions to Problems 1 and 2 in blue book #1, and your solutions to Problems 3-5 in blue book #2 to facilitate grading. You may solve the problems in any order.

**Problem 1.** (20 points)

(i) State the definition of a Cauchy sequence \( \{a_n\}_{n=1}^{\infty} \) of real numbers.

(ii) State the definition that a sequence \( \{a_n\}_{n=1}^{\infty} \) of real numbers is bounded.

(iii) Show directly from the definition of a Cauchy sequence (thus without using theorems that relate these to convergent sequences) that every Cauchy sequence in \( \mathbb{R} \) is bounded.

**Solution.** We say that \( \{a_n\}_{n=1}^{\infty} \) is a Cauchy sequence if given \( \epsilon > 0 \) there exists \( N \in \mathbb{P} \) such that \( n, m \geq N \) implies \( |a_n - a_m| < \epsilon \).

We say that \( \{a_n\}_{n=1}^{\infty} \) is bounded if there exists \( M \in \mathbb{R} \) such that for all \( n \in \mathbb{P} \), \( |a_n| \leq M \).

Suppose \( \{a_n\}_{n=1}^{\infty} \) is bounded. Apply the definition with \( \epsilon = 1 \). Thus, there exists \( N \) such that for \( n, m \geq N \), \( |a_n - a_m| < 1 \). Now use this with \( m = N \). We conclude that for \( n \geq N \), \( |a_n - a_N| < 1 \), so \( |a_n| \leq |a_n - a_N| + |a_N| < |a_N| + 1 \). Now the set \( \{a_n : n < N\} \cup \{a_N + 1\} \) is finite, so there exists \( M \in \mathbb{R} \) such that \( M \geq |a_n| \) for \( n < N \) and \( M \geq |a_N| + 1 \) - one can take \( M \) to be the largest element of this finite set. Then \( n \in \mathbb{P} \) implies that either \( n < N \) or \( n \geq N \), and in either case \( M \geq |a_n| \).

**Problem 2.** (20 points)

(i) State the definition of a series \( \sum_{n=1}^{\infty} a_n \), with \( a_n \in \mathbb{R} \) for each \( n \in \mathbb{P} \), converges.

(ii) Suppose that \( \sum_{n=1}^{\infty} a_n \) is a series with non-negative terms which converges. Let \( \{a_{n_k}\}_{k=1}^{\infty} \) be a subsequence of \( \{a_n\}_{n=1}^{\infty} \). Prove that \( \sum_{k=1}^{\infty} a_{n_k} \) converges and \( \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} a_{n_k} \).

**Solution.**

(i) A series \( \sum_{n=1}^{\infty} a_n \) converges to \( L \in \mathbb{R} \) if the sequence \( \{s_n\}_{n=1}^{\infty} \) of partial sums, \( s_n = \sum_{k=1}^{n} a_k \), converges to \( L \in \mathbb{R} \), i.e. if for all \( \epsilon > 0 \) there exists \( N \in \mathbb{P} \) such that \( n \geq N \) implies \( |s_n - L| < \epsilon \).

(ii) The series \( \sum_{k=1}^{\infty} a_{n_k} \) has non-negative terms, and the partial sums satisfy

\[
\sum_{k=1}^{K} a_{n_k} \leq \sum_{n=1}^{n_K} a_n.
\]

Notice that the right hand side is a partial sum for \( \sum_{k=1}^{\infty} a_n \). As \( \sum_{n=1}^{\infty} a_n \) has non-negative terms, the series of its partial sums is monotone increasing, and as the series converges, its sum is the sup of the partial sums. Thus,

\[
\sum_{k=1}^{K} a_{n_k} \leq \sum_{n=1}^{\infty} a_n.
\]

for all \( K \), i.e. the set of partial sums of \( \sum_{k=1}^{\infty} a_{n_k} \) is bounded. Again, these partial sums form a monotone increasing sequence, thus their sequence converges due to the boundedness, with the sum being the sup of the partial sums. Since \( \sum_{n=1}^{\infty} a_n \) is an
upper bound for the partial sums of this series by what we have shown, \( \sum_{k=1}^{\infty} a_{n_k} \leq \sum_{n=1}^{\infty} a_n \) as desired.

Problem 3. (20 points)

(i) State the definitions of a set being countably infinite, resp. uncountable.

(ii) An accumulation point of a subset of \( \mathbb{R} \) is a point \( x \in \mathbb{R} \) for which there exists a sequence \( \{x_n\}_{n=1}^{\infty} \) in \( X \) such that \( x_n \neq x \) for any \( n \) and such that \( \lim_{n \to \infty} x_n = x \).

Show that if \( a < b \) then any infinite subset \( X \) of \( [a,b] \) has an accumulation point.

(iii) Show that any uncountable subset \( X \) of \( \mathbb{R} \) has an accumulation point.

Solution.

(i) A set \( X \) is countably infinite if there is a bijection from \( X \) to \( \mathbb{N} \). A set is uncountable if it is not countably infinite and is not finite (i.e. is not empty, and there is no bijection from \( X \) to \( \{ k \in \mathbb{N} : k \leq n \} \) for any \( n \in \mathbb{N} \)).

(ii) Suppose \( X \) is an infinite subset of \([a,b]\). Then there is a sequence \( \{x_n\}_{n=1}^{\infty} \) in \( X \) such that \( x_n \neq x_m \) for \( n \neq m \). (The proof is that one constructs the \( x_n \) recursively: there is a point \( x_1 \in X \) since \( X \) is non-empty, and if \( x_1,\ldots,x_n \) have been chosen, there is \( x_{n+1} \in X \) such that \( x_{n+1} \neq x_j \) for \( j \leq n \), for otherwise \( X = \{ x_1,\ldots,x_n \} \) is in bijection with \( \{ k \in \mathbb{N} : k \leq n \} \), and is thus finite.) By the Bolzano-Weierstrass theorem, \( \{x_n\}_{n=1}^{\infty} \) has a convergent subsequence \( \{x_{n_k}\}_{k=1}^{\infty} \) with \( x_{n_k} \to x \in \lim_{n \to \infty} x_n \). If \( x_{n_k} \neq x \) for any \( k \), this shows that \( x \) is an accumulation point of \( X \), and we are done.

Otherwise \( x = x_{n_k} \) for some \( K \), and \( K \) is unique (since \( x_{n_k} \neq x_m \) if \( n \neq m \)), so if we let \( x_k = x_{n_{k+1}} \), \( k \in \mathbb{N} \), \( \{x_k\}_{k=1}^{\infty} \) is a subsequence of \( \{x_{n_k}\}_{k=1}^{\infty} \) and thus converges to \( x \), but in addition, \( x_k \neq x \) for any \( k \) – this sequence is automatically in \( X \), and thus shows that \( x \) is an accumulation point of \( X \) as well.

(iii) If \( X \) is an uncountable subset of \( \mathbb{R} \), we claim that \( X_n = X \cap [-n,n] \) is infinite for some \( n \). Indeed, otherwise each \( X_n \) is finite, and \( X = \bigcup_{n=1}^{\infty} X_n \) is a countable union of finite, thus countable, sets, and so is countable – contradicting its uncountability.

Thus, \( X \) is infinite for some \( n \), and thus by (ii) has an accumulation point \( x \), i.e. there is a sequence \( \{x_k\}_{k=1}^{\infty} \) in \( X_n \) converging to \( x \) with \( x_k \neq x \) for any \( k \) – this sequence is automatically in \( X \), and thus shows that \( x \) is an accumulation point of \( X \) as well.

Problem 4. (20 points) Let \( X \) be the subset of \( \mathbb{R} \) given by

\[
X = \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.
\]

Let \( d \) be the relative metric on \( X \) induced from \( \mathbb{R} \), i.e. for \( x, y \in X \), \( d(x,y) = |x-y| \). Show that a real-valued sequence \( \{a_n\}_{n=1}^{\infty} \) converges to \( L \in \mathbb{R} \) if and only if the function defined on \( X \) by \( f(1/n) = a_n \), \( n \in \mathbb{N} \), \( f(0) = L \), is continuous.

Solution. Suppose first that \( f \) is continuous. Then whenever \( x_n \to x \) in \( X \), \( f(x_n) \to f(x) \) in \( \mathbb{R} \). Applying this with \( x_n = 1/n \), \( x = 0 \), we deduce that \( \lim_{n \to \infty} a_n = L \).

For the converse, suppose that \( f \) is continuous at \( 1/m \) for any \( m \in \mathbb{N} \). Indeed, taking \( \delta < \frac{1}{m} - \frac{1}{m+1} \), \( \frac{1}{m} - \frac{1}{m+1} \), the only point \( x \) in \( X \) with \( d(1/m,x) < \delta \) is \( 1/m \), so given \( \epsilon > 0 \) one can take \( \delta > 0 \) as above and conclude automatically that \( d(1/m,x) < \delta \) implies \( |f(1/m) - f(x)| < \epsilon \). So it remains to consider \( x = 0 \). Then given \( \epsilon > 0 \) there is \( N \in \mathbb{N} \) such that \( n \geq N \) implies \( |a_n - L| < \epsilon \). Let \( \delta = \frac{1}{N} \). Then \( d(x,0) < \delta \), \( x \in X \), implies that either \( x = 0 \) or \( x = 1/n \) with \( n > N \). In either case \( |f(x) - f(0)| < \epsilon \); in the former case the left hand side vanishes, in the latter it is \( |a_n - L| \), and \( n > N \). This completes the proof.

Problem 5. (20 points) Let \( \ell^\infty \) denote the set of bounded real-valued sequences, with the standard metric, \( d(\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}) = \sup \{|a_n - b_n| : n \in \mathbb{N} \} \), and let \( c_0 \) denote the subset of \( \ell^\infty \) consisting of sequences converging to 0. Show that \( c_0 \) is a closed subset of \( \ell^\infty \).

Solution. Let \( d(\{a_n\}, \{b_n\}) = \sup \{|a_n - b_n| \} \) be the standard metric. Suppose that \( \{a^{(k)}_n\}_{n=1}^{\infty} \) is a sequence in \( c_0 \) converging to some \( a \in \ell^\infty \). That is, for any \( \epsilon > 0 \) there is \( K \in \mathbb{N} \) such that \( k \geq K \) implies \( d(a^{(k)}, a) < \epsilon \), i.e. \( \sup_{n\in\mathbb{N}} |a^{(k)}_n - a| < \epsilon \), so \( |a^{(k)}_n - a_n| < \epsilon \) whenever \( k \geq K \) and \( n \in \mathbb{N} \).
We need to show that \( \lim_{n \to \infty} a_n = 0 \), i.e. that for any \( \epsilon' > 0 \) there exists \( N \in \mathbb{P} \) such that \( n \geq N \) implies \( |a_n| < \epsilon' \). To see this, let \( \epsilon = \frac{\epsilon'}{2} \), and apply the argument of the previous paragraph with this \( \epsilon \) to obtain \( K \). Now, with \( k = K \), we have, as \( a_n^{(k)} \in c_0 \), that \( \lim_{n \to \infty} a_n^{(k)} = 0 \), i.e. there is \( N \) such that for \( n \geq N \), \(|a_n^{(k)}| < \epsilon'/2 \). Thus, for \( n \geq N \),

\[
|a_n| \leq |a_n - a_n^{(k)}| + |a_n^{(k)}| < \frac{\epsilon'}{2} + \frac{\epsilon'}{2} = \epsilon',
\]

which completes the proof.