MATH 171: MIDTERM – SOLUTIONS
THURSDAY, MAY 5, 2011

This is a closed book, closed notes, no calculators/computers exam. As usual, \( \mathbb{P} \) denotes the set of positive integers, and \( \mathbb{R}^n \) comes with the Euclidean metric unless otherwise specified.

You may quote any theorem from the textbook or the lecture provided that you are not explicitly asked to prove it, and provided you state the theorem precisely and concisely (make sure to check the hypotheses in writing when you quote a theorem).

There are 5 problems. Solve all of them. Write your solutions to Problems 1 and 2 in blue book #1, and your solutions to Problems 3-5 in blue book #2 to facilitate grading. You may solve the problems in any order.

**Problem 1.** (15 points)

(i) State the definition of the limit of a sequence \( \{a_n\}^{\infty}_{n=1} \) of real numbers.

(ii) State the definition that a sequence \( \{a_n\}^{\infty}_{n=1} \) of real numbers is bounded.

(iii) Suppose that \( \{a_n\} \) is a bounded sequence of real numbers. Let \( b_n = a_n/n \). Show directly from the definition of the limit (i.e. without using limit theorems) that

\[ \lim_{n \to \infty} b_n = 0. \]

**Solution.**

(i) \( L \) is the limit of \( \{a_n\}^{\infty}_{n=1} \) if for all \( \epsilon > 0 \) there exists \( N \in \mathbb{P} \) such that \( n \geq N \implies |a_n - L| < \epsilon \).

(ii) \( \{a_n\}^{\infty}_{n=1} \) is bounded if there exists \( M \geq 0 \) such that for all \( n \in \mathbb{P} \), \( |a_n| \leq M \).

Equivalently, \( \{a_n\}^{\infty}_{n=1} \) is bounded if it is bounded above and below, i.e. there exists \( M_1, M_2 \in \mathbb{R} \) such that \( M_1 \leq a_n \leq M_2 \) for all \( n \in \mathbb{P} \).

(iii) Since \( \{a_n\}^{\infty}_{n=1} \) is bounded, there exists \( M \) such that \( |a_n| \leq M \) for all \( n \). Let \( \epsilon > 0 \) be given. Let \( N \in \mathbb{P} \) be such that \( N > M\epsilon^{-1} \). Then for \( n \geq N \),

\[ |b_n - 0| = |b_n| = |a_n|/n \leq M/N < \epsilon, \]

so \( \lim_{n \to \infty} b_n = 0. \)

**Problem 2.** (20 points) Suppose \( (M, d) \) is a metric space.

(i) State the definition of a subset \( C \) of \( M \) being closed.

(ii) For \( x \in M \) and \( r > 0 \) let \( B_r(x) = \{y \in M : d(y, x) \leq r\} \). Show that \( B_r(x) \) is closed.

(It is called the closed ball of radius \( r \) around \( x \).)

**Solution.**

(i) \( C \) is closed if it contains all of its limit points, i.e. if \( \{x_n\}^{\infty}_{n=1} \) is a sequence in \( C \) and it converges to some \( x \in M \), then \( x \in C \).

(ii) Suppose \( y_n \in B_r(x) \) for all \( n \), and \( \lim_{n \to \infty} y_n = y \in M \). Since the function \( f : M \to \mathbb{R} \), \( f(z) = d(z, x) \) is continuous, \( f(y) = \lim_{n \to \infty} f(y_n) \). But \( f(y_n) = d(y_n, x) \leq r \) for all \( n \), so the limit also satisfies \( d(y, x) = f(y) \leq r \). Thus, \( y \in B_r(x) \), so \( B_r(x) \) is closed.

An alternative proof is as follows: to show that \( B_r(x) \) is closed, it suffices to prove that its complement is open. So suppose \( y \notin B_r(x) \), i.e. \( d(y, x) > r \). Let \( \epsilon = d(y, x) - r > 0 \). We claim that \( B_r(y) \subset (B_r(x))^c = M \setminus B_r(x) \); this implies (by the definition of openness) that \( (B_r(x))^c \) is open. Indeed, if \( z \in B_r(y) \) then by the triangle inequality \( d(x, y) \leq d(x, z) + d(z, y) \), so \( d(z, x) \geq d(x, y) - d(z, y) > d(x, y) - \epsilon = r \), so \( z \notin B_r(x) \), and so \( B_r(y) \subset (B_r(x))^c \).

**Problem 3.** (15 points) Suppose that \( A \neq \emptyset \) is a bounded subset of \( \mathbb{R} \) and has the following property:

\[ x, z \in A \text{ and } x < y < z \Rightarrow y \in A. \]

Let \( a = \inf A \), \( b = \sup A \). Show that

\[ a < y < b \Rightarrow y \in A. \]
Conclude that $A$ must be one of the following intervals: $(a, b)$, $(a, b]$, $[a, b)$, $[a, b]$ (with $[a, b]$ being the only possibility if $a = b$).

Solution. Suppose $a < y < b$. As $y > a$, $y$ is not a lower bound for $A$, i.e. there exists some $x \in A$ such that $x < y$. Similarly, as $y < b$, $y$ is not an upper bound for $A$ so there exists some $z \in A$ such that $y < z$. Thus, $x < y < z$, $x, z \in A$, so $y \in A$. Thus, $(a, b) \subset A$.

Now, if $x < a$ then $x \notin A$ since $a$ is a lower bound for $A$, and similarly if $x > b$ then $x \notin A$ since $b$ is an upper bound for $A$. Thus $(\infty, a) \cup (b, \infty) \subset A^c = \mathbb{R} \setminus A$.

As $\mathbb{R} = (\infty, a) \cup \{a\} \cup (a, b) \cup \{b\} \cup (b, \infty)$, the only question is whether $a$ and $b$ are in $A$; listing the four possibilities (only two if $a = b$) gives the four intervals (only one if $a = b$ as we assumed that $A$ was non-empty).

Problem 4. (25 points) Recall that $\ell^2$ is the vector space of square summable sequences with $\|\{a_n\}\|_{\ell^2} = \sqrt{\sum_{n=1}^{\infty} a_n^2}$, and $\ell^\infty$ is the vector space of bounded sequences with $\|\{a_n\}\|_{\ell^\infty} = \sup_{n \in \mathbb{P}} |a_n|$.

(i) Show that $\ell^2 \subset \ell^\infty$.

(ii) Show that if $a, a^{(k)} \in \ell^2$, $k \in \mathbb{P}$, and $\lim_{k \to \infty} a^{(k)} = a$ in $\ell^2$ then $\lim_{k \to \infty} a^{(k)} = a$ in $\ell^\infty$.

(iii) Give an example of a, $a^{(k)} \in \ell^2$, $k \in \mathbb{P}$ such that $\lim_{k \to \infty} a^{(k)} = a$ in $\ell^\infty$ but $a^{(k)}$ does not converge to $a$ in $\ell^2$.

Solution. (i) Suppose $\{a_n\} \in \ell^2$, so $\sum_{n=1}^{\infty} a_n^2$ converges. Then for any $k$ and for any $N \geq k$, $a_k^2 \leq \sum_{n=1}^{N} a_n^2$, so as $\sum_{n=1}^{\infty} a_n^2$ is the supremum of finite sums, $a_k^2 \leq \sum_{n=1}^{\infty} a_n^2$. Taking square roots, $|a_k| \leq \|\{a_n\}\|_{\ell^2}$. Thus, the sequence $\{a_n\}$ is bounded, with $\|\{a_n\}\|_{\ell^2}$ an upper bound for $\{a_k\} : k \in \mathbb{P}$. In particular,

$$\|\{a_n\}\|_{\ell^\infty} \leq \|\{a_n\}\|_{\ell^2}$$

for the left hand side is the $\textit{least}$ upper bound for $\{a_k\} : k \in \mathbb{P}$.

(ii) Suppose $\lim_{k \to \infty} a^{(k)} = a$ in $\ell^2$ and let $\epsilon > 0$. Then there exists $N$ such that $k \geq N$ implies $\|a^{(k)} - a\|_{\ell^2} < \epsilon$. But $a^{(k)} - a \in \ell^2 \subset \ell^\infty$, so by (1), for $k \geq N$, $\|a^{(k)} - a\|_{\ell^\infty} \leq \|a^{(k)} - a\|_{\ell^2} < \epsilon$. Thus, $\lim_{k \to \infty} a^{(k)} = a$ in $\ell^\infty$.

(iii) Consider the sequence $\{a^{(k)}\}_{k=1}^{\infty}$ where $a^{(n)} = 0$ if $n \neq k$ and $a^{(n)} = k^{-1/2}$ if $n = k$, and let $a$ be the zero sequence: $a_n = 0$ for all $n \in \mathbb{P}$. Note that $a^{(k)} \in \ell^2$ and $a \in \ell^2$ since each of these sequences has finitely many non-zero entries. Then for each $k \in \mathbb{P}$, $\|\{a^{(k)} - a\}_{k=1}^{\infty}\|_{\ell^\infty} = k^{-1/2}$, so $\lim_{k \to \infty} k^{-1/2} = 0$ shows that $\lim_{k \to \infty} a^{(k)} = a$ in $\ell^\infty$.

On the other hand $\|a^{(k)}\|_{\ell^2} = \sqrt{\sum_{n=1}^{k} k^{-1}} = 1$, so $a^{(k)}$ does not converge to $a$ in $\ell^2$ (take $\epsilon = 1$: there exists no $k$ such that $\|a^{(k)} - a\|_{\ell^2} < \epsilon$).

Problem 5. (25 points) If $(M_1, d_1), (M_2, d_2)$ are metric spaces, let $M = M_1 \times M_2$ be the metric space with $d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2)$. We call $d$ the product metric on $M$.

(i) Show that the product metric on $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ is equivalent to the Euclidean metric on $\mathbb{R}^{2n}$, i.e. a set is open in the product metric if and only if it is open in the Euclidean metric on $\mathbb{R}^{2n}$.

(ii) Show that $+: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

(iii) Show that if $(X, \rho)$, $(M_1, d_1)$ and $(M_2, d_2)$ are metric spaces, $f_1 : X \to M_1$ and $f_2 : X \to M_2$ are continuous then $f : X \to M_1 \times M_2$ defined by $f(x) = (f_1(x), f_2(x))$ is continuous.

(iv) With $X$ as in (iii), show that if $f_1 : X \to \mathbb{R}^n$ and $f_2 : X \to \mathbb{R}^n$ are continuous then $f_1 + f_2 : X \to \mathbb{R}^n$, given by $(f_1 + f_2)(x) = f_1(x) + f_2(x)$, is also continuous.

Solution. (i) With $\|\cdot\|_{\mathbb{R}^k}$ denoting the Euclidean norm on $\mathbb{R}^k$, we note that $d_{\mathbb{R}^n \times \mathbb{R}^n}((x_1, x_2), (y_1, y_2)) = \|x_1 - y_1\|_{\mathbb{R}^n} + \|x_2 - y_2\|_{\mathbb{R}^n}$, while $d_{\mathbb{R}^{2n}}((x_1, x_2), (y_1, y_2)) = \sqrt{\|x_1 - y_1\|_{\mathbb{R}^n}^2 + \|x_2 - y_2\|_{\mathbb{R}^n}^2}$.
Thus, \( d_{\mathbb{R}^n \times \mathbb{R}^n}((x_1, x_2), (y_1, y_2)) < \epsilon/2 \) implies that \( \|x_1 - y_1\|_{\mathbb{R}^n} < \epsilon/2 \) and \( \|x_2 - y_2\|_{\mathbb{R}^n} < \epsilon/2 \), so \( d_{\mathbb{R}^n \times \mathbb{R}^n}((x_1, x_2), (y_1, y_2)) < \sqrt{\epsilon^2/2} \leq \epsilon \), so \( B_{\epsilon/2}^{\mathbb{R}^n \times \mathbb{R}^n}(x) \subset B_{\epsilon}^{\mathbb{R}^n}(x) \).

Similarly, \( d_{\mathbb{R}^n \times \mathbb{R}^n}((x_1, x_2), (y_1, y_2)) < \epsilon/2 \) implies \( \|x_1 - y_1\|_{\mathbb{R}^n} < \epsilon/2 \) and \( \|x_2 - y_2\|_{\mathbb{R}^n} < \epsilon/2 \), so \( d_{\mathbb{R}^n \times \mathbb{R}^n}((x_1, x_2), (y_1, y_2)) < \epsilon \), hence \( B_{\epsilon/2}^{\mathbb{R}^n}(x) \subset B_{\epsilon}^{\mathbb{R}^n \times \mathbb{R}^n}(x) \).

Now, if \( O \) is open in the product metric and \( x \in O \) then there exists \( \epsilon > 0 \) such that \( B_{\epsilon}^{\mathbb{R}^n \times \mathbb{R}^n}(x) \subset O \) and thus \( B_{\epsilon/2}^{\mathbb{R}^2}(x) \subset B_{\epsilon}^{\mathbb{R}^n \times \mathbb{R}^n}(x) \subset O \), so \( O \) is open in the Euclidean metric. Conversely, if \( O \) is open in the Euclidean metric then there exists \( \epsilon > 0 \) such that \( B_{\epsilon}^{\mathbb{R}^n}(x) \subset O \) and thus \( B_{\epsilon/2}^{\mathbb{R}^n}(x) \subset B_{\epsilon}^{\mathbb{R}^n \times \mathbb{R}^n}(x) \subset O \), so \( O \) is open in the product metric, completing the proof.

A different proof is as follows: it is equivalent to show that the closed sets are the same, and thus to show that convergent sequences are the same. A sequence \( \{(x_1^{(n)}, x_2^{(n)})\} \) converges in the product metric to \( (x_1, x_2) \) if and only if \( \{x_j^{(n)}\} \) converges to \( x_j \) in \( \mathbb{R}^n \) for \( j = 1, 2 \). But this happens if and only if all the coordinates \( x_j^{(n)}, \) \( k = 1, \ldots, n \) converges to \( x_{j,k} \). On the other hand, \( \{(x_1^{(n)}, x_2^{(n)})\} \) converges in the \( \mathbb{R}^2 \) metric if and only if all the coordinates converge, i.e. if and only if \( x_j^{(n)}, k = 1, \ldots, n \) converges to \( x_{j,k} \). Thus, the two notions of convergence are the same.

(ii) Suppose that \( (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n \) and \( \epsilon > 0 \). If \( (y_1, y_2) \in \mathbb{R}^n \) then by the triangle inequality in \( \mathbb{R}^n \),

\[
\|(y_1 + y_2) - (x_1 + x_2)\|_{\mathbb{R}^n} = \|(y_1 - x_1) + (y_2 - x_2)\|_{\mathbb{R}^n} \leq \|y_1 - x_1\|_{\mathbb{R}^n} + \|y_2 - x_2\|_{\mathbb{R}^n}.
\]

But

\[
d_{\mathbb{R}^n \times \mathbb{R}^n}((y_1, y_2), (x_1, x_2)) = d_{\mathbb{R}^n}(y_1, x_1) + d_{\mathbb{R}^n}(y_2, x_2) = \|y_1 - x_1\|_{\mathbb{R}^n} + \|y_2 - x_2\|_{\mathbb{R}^n},
\]

so taking \( \delta = \epsilon \), \( d_{\mathbb{R}^n \times \mathbb{R}^n}((y_1, y_2), (x_1, x_2)) < \delta \) implies \( d_{\mathbb{R}^n}(+(y_1, y_2), +(x_1, x_2)) < \epsilon \), proving the desired continuity.

(iii) Let \( x \in X \) and \( \epsilon > 0 \). Then, by the definition of continuity of \( f_j \), there exists \( \delta_j > 0 \), \( j = 1, 2 \) such that \( \rho(y, x) < \delta_j \) implies \( d_j(f_j(y), f_j(x)) < \epsilon/2 \). Let \( \delta = \min(\delta_1, \delta_2) > 0 \). Thus \( \rho(y, x) < \delta \) implies \( d(f(y), f(x)) = d_1(f_1(y), f_1(x)) + d_2(f_2(y), f_2(x)) < \epsilon \), giving the desired continuity.

(iv) We have \( f_1 + f_2 = + \circ f \) where \( f = (f_1, f_2) \) as in (iii) with \( M_f = \mathbb{R}^n \) and \( + : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) as in (ii). Since the composition of continuous maps is continuous, \( f_1 + f_2 \) is continuous.