MATH 131P: PRACTICE FINAL
DECEMBER 12, 2012

This is a closed book, closed notes, no calculators/computers exam.
There are 6 problems. Write your solutions to Problems 1-3 in blue book #1, and your
solutions to Problems 4-6 in blue book #2 to facilitate grading. You may solve the problems in
any order. Total score: 150 points. Problem 4(iv) is extra credit only; do it only if you are done
with the rest of the exam.

Problem 1. (20 points) Solve the PDE
\[ u_x + 2x u_y = 2u, \quad u(0, y) = y^2, \]
and sketch the characteristics of the PDE.

Problem 2. (20 points) Consider the PDE
(i) \( u_{xx} - u_{xy} - 2u_{yy} = 0, \)
(ii) \( u_{xx} + 3u_{xy} + 3u_{yy} = 0. \)
What is the type of these PDE? One of the PDE is hyperbolic. Find its general solution.

Problem 3. Consider Laplace’s equation on the rectangle \( R: \)
\[ u_{xx} + u_{yy} = 0 \text{ on } R = (0, a) \times (0, b), \]
with boundary conditions \( u(0, y) = 0, \ u(a, y) = y, \ u_x(x, 0) = 0, \ u_y(x, b) = 0. \)
(i) (7 points) Interpret the PDE and the boundary conditions: if \( u \) is the steady state
temperature on a metal plate, what is imposed at the various edges of the plate? Draw
a picture.
(ii) (12 points) Use separation of variables to solve the PDE. Find all coefficients in your
expansion explicitly.

Problem 4. Consider the wave equation \( u_{tt} = c^2 u_{xx} \) on the half-line, i.e. on \((0, \infty), \times \mathbb{R}, \)
with homogeneous Neumann boundary condition \( u_x(0, t) = 0, \) and with initial conditions \( u(x, 0) = f(x) \) and \( u_t(x, 0) = g(x) \) for \( x \geq 0. \)
(i) (12 points) Find \( u \) in terms of \( f, g. \) Make your answer as explicit and simplified as
possible.
(ii) (6 points) What are the time symmetric solutions \( u, \) i.e. solutions \( u \) such that \( u(x, t) = \\
u(x, -t) \) for all \( x, t? \)
(iii) (12 points) Suppose that \( c = 1, \) \( g \equiv 0, \) and \( f(x) = 1 \) for \( 1 < x < 2, \) \( f(x) = 0 \) for \( x > 1 \)
and for \( 0 < x < 1. \) Find \( u(0,t) \) explicitly for \( t = 1/2, \) \( t = 1, \) \( t = 3/2 \) and \( t = 2, \) and
sketch the profiles.
(iv) (Bonus: 15 points) Assume \( u \) is a solution of the wave equation with Neumann boundary
condition, and on finite intervals \([-T, T], \) \( u \) vanishes for large \( x. \) The kinetic energy of the
wave at time \( t \) is (up to a constant factor) \( K(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_t(x, t)^2 \, dx, \) while the potential
energy is \( P(t) = \frac{1}{2} \int_{-\infty}^{\infty} c^2 u_x(x, t)^2 \, dx. \) Let \( E(t) = K(t) + P(t) \) be the total energy of the
wave, and show that \( \frac{dE}{dt} = 0, \) hence \( E \) is constant (independent of \( t). \)
You may use in any part of the problem that if \( v \) solves \( v_{tt} - c^2 v_{xx} = 0 \) on \( \mathbb{R} \times \mathbb{R}, \) then
\[ v(x, t) = \frac{v(x - ct, 0) + v(x + ct, 0)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} u_t(x', 0) \, dx'. \]

Problem 5. Consider Laplace’s equation \( \Delta u = 0 \) on the annular region \( A(1, 2) \) with inner and
outer radii \( r_1 = 1 \) and \( r_2 = 2, \) namely
\[ \Delta u = 0, \quad u(r_1, \theta) = f(\theta), \quad u(r_2, \theta) = g(\theta), \quad \theta \in [0, 2\pi]. \]
f, g given functions. Recall that \( \Delta u = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} \) in polar coordinates.
(i) (13 points) If \( f(\theta) = a \) and \( g(\theta) = b \) are constants, find \( u \), and sketch the graph of \( u \) in each of the cases \( a > b \), \( a = b \) and \( a < b \). What is the maximum value of \( u \) on the annulus?

(ii) (12 points) If \( f(\theta) = \sin(2\theta) \) and \( g(\theta) = \sin(3\theta) \), find \( u \).

**Problem 6.** The purpose of this problem is to solve \( \Delta u = f \) on \( \mathbb{R}^3 \) when \( f(\vec{x}) \) vanishes for large \( |\vec{x}| \), more precisely to find the decaying solution \( u \) of this equation.

(i) (7 points) Fourier transform the PDE to obtain the solution \( u \) as an inverse Fourier transform.

(ii) (5 points) Rewrite the inverse Fourier transform so that \( u \) is a convolution of \( f \) with a function which is the inverse Fourier transform of an explicit function.

(iii) (8 points) Calculate the Fourier transform of the function \( g(\vec{x}) = \frac{1}{|\vec{x}|} \) by considering \( g_\epsilon(\vec{x}) = e^{-\epsilon |\vec{x}|} \frac{1}{|\vec{x}|} \) first for \( \epsilon > 0 \), computing the Fourier transform of \( g_\epsilon \), and letting \( \epsilon \to 0 \).

(iv) (5 points) Write \( u \) as an explicit convolution without any (inverse) Fourier transforms. Recall that the volume integral in \( \mathbb{R}^3 \) takes the form

\[
\int_{\mathbb{R}^3} h(\vec{x}) \, d\vec{x} = \int_0^\infty \int_0^\pi \int_0^{2\pi} h(r, \theta, \phi) r^2 \sin \theta \, d\phi \, d\theta \, dr
\]

in spherical coordinates \((r, \theta, \phi)\) where \( x = r \sin \theta \cos \phi, \ y = r \sin \theta \sin \phi, \ z = r \cos \theta \).