3.) Define \( c_j = \sqrt{\mathbf{a}_j}, \ d_j = \frac{1}{\sqrt{\mathbf{a}_j}} \mathbf{b}_j \), for \( j = 1, 2, \ldots, n \). Let \( \bar{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \) and \( \bar{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} \) be vectors in \( \mathbb{R}^n \). Then the Cauchy-Schwarz inequality on Euclidean \( n \)-space gives

\[
\left( \sum_{j=1}^{n} c_j d_j \right)^2 \leq \left( \sum_{j=1}^{n} c_j^2 \right) \left( \sum_{j=1}^{n} d_j^2 \right).
\]

On writing \( c_j \) and \( d_j \) in terms of \( a_j \) and \( b_j \) in the above equation, we have

\[
\left( \sum_{j=1}^{n} a_j b_j \right)^2 \leq \left( \sum_{j=1}^{n} j a_j^2 \right) \left( \sum_{j=1}^{n} j b_j^2 \right)
\]
as desired.

5.) No such inner product exists. We prove this by showing that the parallelogram equality is violated for the vectors \( \mathbf{u} = (1, 0) \) and \( \mathbf{v} = (0, 1) \). We have \( \|\mathbf{u} + \mathbf{v}\| = \|(1, 1)\| = 2 \) and \( \|\mathbf{u} - \mathbf{v}\| = \|(1, -1)\| = 2 \) while \( \|\mathbf{u}\| = \|\mathbf{v}\| = 1 \). Thus \( \|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 8 \neq 2(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2) = 4 \).

6.) We have \( \|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2 = \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle - \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle \), which, after expanding yields, \( (\langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle) - (\langle \mathbf{u}, \mathbf{u} \rangle - \langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle) \). Collecting terms, we have that

\[
\|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2 = 2\langle \mathbf{u}, \mathbf{v} \rangle + 2\langle \mathbf{v}, \mathbf{u} \rangle = 4\langle \mathbf{u}, \mathbf{v} \rangle,
\]
and, on dividing by 4, we have the result.

7.) Expanding the right hand side,

\[
\sum_{p=1}^{4} i^p \|\mathbf{u} + i^p \mathbf{v}\|^2 = \sum_{p=1}^{4} i^p \langle \mathbf{u} + i^p \mathbf{v}, \mathbf{u} + i^p \mathbf{v} \rangle = \sum_{p=1}^{4} i^p (\langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, i^p \mathbf{v} \rangle + \langle i^p \mathbf{v}, \mathbf{u} \rangle + \langle i^p \mathbf{v}, i^p \mathbf{v} \rangle).
\]

Now \( \langle \mathbf{u}, i^p \mathbf{v} \rangle = i^{-p} \langle \mathbf{u}, \mathbf{v} \rangle \) and \( \langle i^p \mathbf{v}, \mathbf{u} \rangle = i^p \langle \mathbf{v}, \mathbf{u} \rangle \), while \( \langle i^p \mathbf{v}, i^p \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{v} \rangle \). Our sum can thus be split

\[
4RHS = \langle \mathbf{u}, \mathbf{u} \rangle \sum_{p=1}^{4} i^p + \langle \mathbf{u}, \mathbf{v} \rangle \sum_{p=1}^{4} 1 + \langle \mathbf{v}, \mathbf{u} \rangle \sum_{p=1}^{4} i^{2p} + \langle \mathbf{v}, \mathbf{v} \rangle \sum_{p=1}^{4} i^p = 4\langle \mathbf{u}, \mathbf{v} \rangle
\]
since \( \sum_{p=1}^{4} i^p = \sum_{p=1}^{4} i^{2p} = 0 \). On dividing by 4 we have the result.
9.) We have \( \int_{-\pi}^{\pi} \left( \frac{1}{\sqrt{2\pi}} \right)^2 = 1 \), so \( \left\| \frac{1}{\sqrt{2\pi}} \right\| = 1 \). Also, \( \sin k \left( x + \frac{\pi}{2k} \right) = \cos kx \), so

\[
\int_{-\pi}^{\pi} \cos^2 kx \, dx = \int_{-\pi}^{\pi} \sin^2 k \left( x + \frac{\pi}{2k} \right) \, dx = \int_{-\pi}^{\pi+\frac{\pi}{2k}} \sin^2 kx \, dx = \int_{-\pi}^{\pi} \sin^2 kx \, dx,
\]

where the last equality holds because \( \sin^2 kx \) is \( 2\pi \) periodic (actually the period is \( \frac{\pi}{k} \)). Thus

\[
\int_{-\pi}^{\pi} \left( \frac{\cos kx}{\sqrt{\pi}} \right)^2 = \int_{-\pi}^{\pi} \left( \frac{\sin kx}{\sqrt{\pi}} \right)^2 = 1.
\]

But \( \cos^2 kx + \sin^2 kx = 1 \), so

\[
2 = \int_{-\pi}^{\pi} \left[ \left( \frac{\cos kx}{\sqrt{\pi}} \right)^2 + \left( \frac{\sin kx}{\sqrt{\pi}} \right)^2 \right] \, dx
\]

and we get \( \int_{-\pi}^{\pi} \left( \frac{\cos kx}{\sqrt{\pi}} \right)^2 = \int_{-\pi}^{\pi} \left( \frac{\sin kx}{\sqrt{\pi}} \right)^2 = 1 \). Together, our calculations show that \( \frac{\sin kx}{\sqrt{\pi}} \), and \( \frac{\cos kx}{\sqrt{\pi}} \) have norm \( 1 \) for \( k = 1, 2, \ldots, n \).

It remains to show that pairs of distinct functions in the list are orthogonal. To prove orthogonality with 1, observe that \( \int_{-\pi}^{\pi} \sin kx \, dx = 0 \) since \( \sin \) is odd. Since \( \cos kx \) is just a phase shift from \( \sin kx \), this proves that \( \int_{-\pi}^{\pi} \cos kx \, dx = 0 \) as well. The same reasoning shows that \( \int_{-\pi}^{\pi} \sin ax \cos bx \, dx = 0 \), since this is odd as well. The remaining integrals to consider have the form \( \int_{-\pi}^{\pi} \sin ax \sin bx \) and \( \int_{-\pi}^{\pi} \cos ax \cos bx \) for \( a \neq b \). But now, \( \cos ax \cos bx + \sin ax \sin bx = \cos(ax - bx) \) and \( \cos ax \cos bx - \sin ax \sin bx = \cos(ax + bx) \). For \( a \neq b \) we have both \( \int_{-\pi}^{\pi} \cos(ax - bx) = 0 \) and \( \int_{-\pi}^{\pi} \cos(ax + bx) = 0 \), so that

\[
\int_{-\pi}^{\pi} \cos ax \cos bx \, dx = \frac{1}{2} \left( \int_{-\pi}^{\pi} \cos(ax + bx) + \cos(ax - bx) \, dx \right) = 0
\]

\[
\int_{-\pi}^{\pi} \sin ax \sin bx \, dx = \frac{1}{2} \left( \int_{-\pi}^{\pi} \cos(ax - bx) - \cos(ax + bx) \, dx \right) = 0.
\]

Thus the entire sequence is orthogonal.

10.) We can put \( e_1 = 1 \) since this has norm 1. Then \( x - \langle x, e_1 \rangle e_1 = x - \int_0^1 x = x - 1/2 \). This has norm

\[
\left\| x - 1/2 \right\|^2 = \int_0^1 x^2 - x + 1/4 = 1/12
\]

so we put \( e_2 = \sqrt{12}(x - 1/2) \). Finally,

\[
x^2 - \langle x^2, e_1 \rangle e_1 - \langle x^2, e_2 \rangle e_2 = x^2 - \int_0^1 x^2 - \sqrt{12}(x - 1/2) \int_0^1 \sqrt{12}(x^3 - 1/2x^2)
\]

\[= x^2 - 1/3 - 12(x - 1/2) (1/4 - 1/6) = x^2 - x + 1/6.\]
This has norm

\[ \|x^2 - x + 1/6\|^2 = \int_0^1 x^4 - 2x^3 + 4/3x^2 - 1/3x + 1/36 = 1/180. \]

Thus \( e_3 = 6\sqrt{3}(x^2 - x + 1/6) \).

17.) Given \( P^2 = P \), we show that \( P \) is the orthogonal projection onto range \( P \). Given \( v \in V \) we can write \( v = u + w \) with \( u \in \text{range } P, w \in (\text{range } P)^\perp \). We need to show that \( Pv = u \).

We know that \( Pv = Pu + Pw \). Observe that \( u \in \text{range } P \) implies there exists \( x \) with \( Px = u \). Then \( Pu = P^2x = Px = u \). We claim that \( (\text{range } P)^\perp = \text{null } P \) so that \( Pw = 0 \). Indeed, the condition each vector of null \( P \) is orthogonal to each vector of range \( P \) implies that \( \text{null } P \subset (\text{range } P)^\perp \). But \( \dim \text{null } P = \dim V - \dim \text{range } P \) by Rank-Nullity and \( \dim (\text{range } P)^\perp = \dim V - \dim \text{range } P \) since \( V = \text{range } P \oplus (\text{range } P)^\perp \). Thus null \( P \) and \( (\text{range } P)^\perp \) have the same dimension, so that they are actually equal. This completes our proof, since we’ve shown \( Pv = Pu + Pw = u + 0 \).

20.) Suppose \( U \) and \( U^\perp \) are invariant under \( T \). Write \( v \in V \) as \( v = u + w \) with \( u \in U, w \in U^\perp \). Then \( P_TTv = P_T(Tu + Tw) \) and \( Tu \in U, Tw \in W \) implies \( P_T(Tu + Tw) = Tu \). Meanwhile, \( TP_U(u + w) = Tu \) so \( TP_Uv = P_TTv \) for every \( v \in V \).

Now suppose that \( TR_U = R_UT \). Take \( u \in U \). Then \( Tu = TP_Uu = R_UTu \) which is an element of \( U \) since \( \text{range } P_U = U \). Thus \( Tu \) is contained in \( U \). Now take \( w \in U^\perp \). We have \( P_UTw = TP_Uw = 0 \). It follows that when we write \( Tw = u' + w' \) with \( u' \in U, w' \in U^\perp \) we have \( u' = 0 \). Thus \( Tw \in U^\perp \) so \( U^\perp \) is also \( T \)-invariant.

22.) Since we are given that \( p(0) = p'(0) = 0 \), \( p \) is a polynomial without constant or linear term. This condition is satified if and only if \( p \) lies in the subspace of \( P_3(\mathbb{R}) \) spanned by \( (x^2, x^3) \). The quantity to be minimized is \( \|2 + 3x - p(x)\|^2 \) where the norm is the one inherited from the inner product \( \langle \cdot, \cdot \rangle \) on \( P_3(\mathbb{R}) \) given by

\[ \langle P, Q \rangle = \int_0^1 P(x)Q(x) \, dx. \]

By proposition 6.36, this minimization is achieved exactly when \( p(x) \) is the orthogonal projection of \( 2 + 3x \) onto the span of \( (x^2, x^3) \).

To calculate this projection, we first find an orthonormal basis for span \( (x^2, x^3) \) with respect to \( \langle \cdot, \cdot \rangle \). We have \( \|x^2\|^2 = \int_0^1 x^4 = 1/5 \) so we choose \( e_1 = \sqrt{5}x^2 \). Then \( x^3 - \langle x^3, e_1 \rangle e_1 = x^3 - 5x^2 \int_0^1 x^5 = x^3 - 5/6x^2 \). This has squared norm

\[ \|x^3 - 5/6x^2\|^2 = \int_0^1 x^6 - 5/3x^5 + 25/36x^4 = 1/252. \]
Thus we set $e_3 = 6\sqrt{7}(x^3 - 5/6x^2)$.

The best choice for $p(x)$ is then given by $p(x) = (2 + 3x, e_1)\langle e_1 + (2 + 3x, e_2)\rangle e_2$, i.e.

$$p(x) = \left(\int_0^1 2\sqrt{5}x^2 + 3\sqrt{5}x^3 \right) \left(\sqrt{5}x^2 + \int_0^1 6\sqrt{7}(2 + 3x)(x^3 - 5/6x^2) \right) 6\sqrt{7}(x^3 - 5/6x^2)$$

$$= 85/12x^2 - 203/10(x^3 - 5/6x^2) = -203/10x^3 + 24x^2.$$

Extra Problem 1:

We first show that $V$ is a vector space. Given $(u_1, w_1)$, $(u_2, w_2) \in V$ we have $(u_1, w_1) + (u_2, w_2) = (u_1 + u_2, w_1 + w_2) = (u_2 + u_1, w_2 + w_1)$ since both $U$ and $W$ are commutative. But then $(u_1, w_1) + (u_2, w_2) = (u_2 + u_1, w_2 + w_1) = (u_2, w_2) + (u_1, w_1)$ so $V$ is commutative.

Given a third vector $(u_3, w_3) \in V$, then $((u_1, w_1) + (u_2, w_2)) + (u_3, w_3) = (u_1 + u_2, w_1 + w_2) + (u_3, w_3) = (u_1 + u_2 + u_3, w_1 + w_2 + w_3) = (u_1 + (u_2 + u_3), w_1 + (w_2 + w_3)) = (u_1, w_1) + (u_2 + u_3, w_2 + w_3) = (u_1, w_1) + ((u_2, w_2) + (u_3, w_3))$ by invoking the associativity of both $U$ and $W$. This proves addition on $V$ is associative.

To show scalar multiplication on $V$ is associative, take $a, b$ scalars and $(u, w) \in V$. Then $(ab)(u, w) = (ab, wv) = a(bu, bw) = a(b(u, w))$.

The additive identity is $(0, 0)$, where the first 0 is the identity in $U$ and the second 0 is the identity in $W$. Indeed, $(u_1, w_1) + (0, 0) = (u_1 + 0, w_1 + 0) = (u_1, w_1)$.

For the additive inverse to $(u_1, w_1)$ we may choose $(-u_1, -w_1)$ where $-u_1$ is the additive inverse of $u_1$ in $U$, and $-w_1$ is the additive inverse of $w_1$ in $W$. We have $(u_1, w_1) + (-u_1, -w_1) = (u_1 - u_1, w_1 - w_1) = (0, 0)$ which is the additive identity we chose for $V$.

We calculate $1 \cdot (u_1, w_1) = (1 \cdot u_1, 1 \cdot w_1) = (u_1, w_1)$ so 1 is the multiplicative identity.

Also $a((u_1, w_1) + (u_2, w_2)) = a(u_1 + u_2, w_1 + w_2) = (a(u_1 + u_2), a(w_1 + w_2)) = (au_1 + aw_1, au_2 + aw_2) = (au_1 + aw_1 + aw_2) = a(u_1, w_1) + a(u_2, w_2)$. Similarly, $(a + b)(u_1, w_1) = ((a + b)u_1, (a + b)w_1) = (au_1 + bu_1, aw_1 + bw_1) = (au_1, aw_1) + (bu_1, bw_1) = a(u_1, w_1) + b(u_1, w_1)$, so that the distributive properties hold for $V$. We have used in these two calculations the distributive properties from $U$ and $W$.

Together the above calculations show that $V$ is a vector space.

To show that $((u_1, 0), ..., (u_m, 0), (0, w_1), ..., (0, w_k))$ forms a basis for $V$, we first check that it spans $V$. Take $(u, w) \in V$. Then we may write $u = a_1u_1 + ... + a_mu_m$ for some $a_1, ..., a_m$ since the $u_i$ span $U$. Similarly we may write $w = b_1w_1 + ... + b_kw_k$ since the $w_j$ span $W$. But then $a_1(u_1, 0) + ... + a_m(u_m, 0) + b_1(0, w_1) + ... + b_k(0, w_k) = (a_1u_1 + ... + a_mu_m, b_1w_1 + ... + b_kw_k) = (u, w).$ This proves that $((u_1, 0), ..., (0, w_k))$ spans $V$. To check that the list is linearly independent, suppose that there exist constants $c_1, ..., c_m, d_1, ..., d_k$ so that $c_1(u_1, 0) + ... + c_m(u_m, 0) + d_1(0, w_1) + ... + d_k(0, w_k) = (0, 0).$ Then $(c_1u_1 + ... + c_mu_m, d_1w_1 + ... + d_kw_k) = (0, 0)$ and in particular, $c_1u_1 + ... + c_mu_m = 0$ in $U$ and $d_1w_1 + ... + d_kw_k = 0$ in $W$. But then the linear independence of the $u_i$ and $w_j$ implies $c_1 = ... = c_m = d_1 = ... = d_k = 0$. This proves that $((u_1, 0), ..., (0, w_k))$ is independent. Thus it forms a basis.

To prove that $\ell \in \mathcal{L}(V, X)$ is an isomorphism we first show that it is linear, and then show that it is a bijection. Take scalar $a$ and two vectors $(u, w)$, and $(u', w')$ in $V$. Then
\[ \iota(a(u, w) + (u', w')) = \iota(au + u', aw + w') = au + u' + aw + w' = a(u + w) + (u' + w') = a\iota(u, w) + \iota(u', w'). \] This proves that \( \iota \) is linear.

To show that \( \iota \) is a bijection, observe that \( X \) is the set of vectors \( \{ z \in \mathbb{Z} | z = u + w; u \in U, w \in W \} \) so \( \iota \) is a genuine map from \( V \) to \( X \) (that is, its image is indeed contained in \( X \)). Moreover, it is onto since for \( x \in X \) we have \( x = u + w \) for some \( u \in U, w \in W \) and \( x = \iota(u, w) \). Furthermore, it is injective, since \( \iota(u_1, w_1) = \iota(u_2, w_2) \) implies \( u_1 + w_1 = u_2 + w_2 \), or \( u_1 - u_2 = w_2 - w_1 \). The left side here is a member of \( U \) and the right side of \( W \), so each is in the intersection. Since \( U \cap W = \{0\} \) we have \( u_1 - u_2 = w_2 - w_1 = 0 \), that is, \( u_1 = u_2, w_1 = w_2 \), so \( (u_1, w_1) = (u_2, w_2) \).

Extra Problem 2:

(1) To check that \( W \) has commutative and associative addition, additive identity and additive inverses is the same set of calculations as in problem 1. We check that \( 1 + 0i \) is the multiplicative identity on \( W \). We have \( (1 + 0i)(u, w) = 1(u, w) + 0(-w, u) = (u, w) + (0(-w), 0u) = (u, w) \) where this calculation uses the fact that \( 1 \) is the multiplicative identity on \( V \oplus V \) and that \( 0 \cdot v = 0 \) for all \( v \in V \).

Next we show that scalar multiplication by complex numbers is associative. For \( a, b, a', b' \) real numbers and \( (u, v) \in W \) we have

\[
((a + bi)(a' + b'i))(u, v) = ((aa' - bb') + (ab' + a'b)i)(u, v)
= (aa' - bb')(u, v) + (ab' + a'b)(-v, u)
= (aa'u - bb'u - ab'v - a'bv, aa'v - bb'v + ab'u + a'bu).
\]

Meanwhile,

\[
(a + bi)((a' + b'i)(u, v)) = (a + bi)(a'(u, v) + b'(-v, u))
= (a + bi)(a'u - b'v, a'v + b'u)
= a(a'u - b'v, a'v + b'u) + b(-a'v - b'u, a'u - b'v)
= (aa'u - bb'u - ab'v - a'bv, aa'v - bb'v + ab'u + a'bu).
\]

Since these two equations are equal, we have that multiplication by complex scalars is associative.

It remains to prove the distributive properties. Take \((u_1, w_1), (u_2, w_2) \in W \) and \( a, b \in \mathbb{R} \). Then

\[
(a + bi)((u_1, w_1) + (u_2, w_2)) = (a + bi)(u_1 + u_2, w_1 + w_2)
= a(u_1 + u_2, w_1 + w_2) + b(-(w_1 + w_2), u_1 + u_2)
= (au_1 - bw_1 + au_2 - bw_2, aw_1 + bu_1 + aw_2 + bu_2)
= ((au_1, aw_1) + (-bw_1, bu_1)) + ((au_2, aw_2) + (-bw_2, bu_2))
= (a + bi)(u_1, w_1) + (a + bi)(u_2, w_2).
\]
Now take a second pair of real numbers \( a', b' \). Then
\[
((a + b) + (a' + b'i))(u, w) = ((a + a') + (b + b')i)(u, w)
= (a + a')(u, w) + (b + b')(-w, u)
= a(u, w) + a'(u, w) + b(-w, u) + b'(-w, u)
= (a(u, w) + b(-w, u)) + (a'(u, w) + b'(-w, u))
= (a + bi)(u, w) + (a' + b'i)(u, w).
\]

These two calculations prove that scalar multiplication by complex numbers distributes over addition, and together with the previous facts, this shows that \( W \) is a vector space over the complex numbers.

(2) To show that \( (e_1, 0), ..., (e_n, 0) \) forms a basis for \( W \) we first show that this list spans, and then show that it is linearly independent. Take \( (u, w) \in W \). Then since \( (e_1, ..., e_n) \) is a basis for \( V \) and \( u, w \in V \) we can find real scalars \( a_1, ..., a_n \) and \( b_1, ..., b_n \) so that \( u = a_1e_1 + ... + a_ne_n \) and \( w = b_1e_1 + ... + b_ne_n \). But then
\[
(a_1 + b_1i)(e_1, 0) + ... + (a_n + b_ni)(e_n, 0)
= (a_1e_1 - b_10, a_10 + b_1e_1) + ... + (a_ne_n - b_n0, a_n0 + b_ne_n)
= (a_1e_1 + ... + a_ne_n, b_1e_1 + ... + b_ne_n) = (u, w).
\]

Hence \( (e_1, 0), ..., (e_n, 0) \) spans all of \( W \). To show that this list is linearly independent, suppose there exist real numbers \( c_1, ..., c_n, d_1, ..., d_n \) with \( \sum_{k=1}^n (c_k + d_ki)(e_k, 0) = (0, 0) \). Then \( \sum_{k=1}^n (c_ke_k, d_ke_k) = (0, 0) \), so in \( V \) we have that \( \sum_{k=1}^n c_ke_k = 0, \sum_{k=1}^n d_ke_k = 0 \). But \( (e_1, ..., e_n) \) is linearly independent in \( V \), so we must have \( c_1 = ... = c_n = d_1 = ... = d_n = 0 \). Hence \( (e_1, 0), ..., (e_n, 0) \) is independent over \( W \). Thus it is a basis.

(3) Since \( T \) maps \( V \) to \( V \), we have \( S(v_1, v_2) = (Tv_1, Tv_2) \in W \) so \( S \) maps \( W \) to \( W \). We show that it is linear. Take \( (u_1, w_1), (u_2, w_2) \in W \) and \( a, b \in \mathbb{R} \). Then
\[
S((u_1, w_1) + (u_2, w_2)) = S(u_1 + u_2, w_1 + w_2)
= (T(u_1 + u_2), T(w_1 + w_2))
= (Tu_1 + Tu_2, Tw_1 + Tw_2)
= (Tu_1, Tw_1) + (Tu_2, Tw_2) = S(u_1, w_1) + S(u_2, w_2).
\]

Also,
\[
S((a + bi)(u_1, w_1)) = S(au_1 - bw_1, aw_1 + bu_1)
= (T(au_1 - bw_1), T(aw_1 + bu_1))
= (aTu_1 - bTw_1, aTw_1 + bTu_1)
= (a + bi)(Tu_1, Tw_1) = (a + bi)S(u_1, w_1)
\]

Together the last two equations show that \( S \) is linear.
Thus we've checked the hint.

Suppose \( \lambda \) is a real eigenvalue of \( \mathbf{S} \) with associated non-zero eigenvector \( \mathbf{v} \). Then \( \mathbf{S}(v,0) = (T v, 0) = (\lambda v, 0) = (\lambda + 0i)(v,0) \), so \( \lambda \) is also an eigenvalue of \( \mathbf{S} \). Suppose instead that \( \lambda \) is a real eigenvalue of \( \mathbf{S} \) with non-zero eigenvector \( (u,w) \). Then \( \mathbf{S}(u,w) = (T u, T w) = (\lambda + 0i)(u,w) = (\lambda u, \lambda w) \). We must have one of \( u, w \) non-zero, say \( u \neq 0 \). Then \( T u = \lambda u \) so \( \lambda \) is an eigenvalue of \( T \).

Before considering the case of complex eigenvalues, we prove the claims made in the hint regarding the conjugation map \( \mathbf{c} \). We have \( \mathbf{c}(\mathbf{S}(v_1, v_2)) = \mathbf{c}(T v_1, T v_2) = (T v_1, -T v_2) \) and \( \mathbf{S}(\mathbf{c}(v_1, v_2)) = \mathbf{S}(v_1, -v_2) = (T v_1, T(-v_2)) = (T v_1, -T v_2) \). Thus indeed, \( \mathbf{c}(\mathbf{S}(v_1, v_2)) = \mathbf{S}(\mathbf{c}(v_1, v_2)) \). Also, for \( \lambda = a + bi \), \( a, b \in \mathbb{R} \), we calculate \( \mathbf{c}(\lambda (v_1, v_2)) = \mathbf{c}((a + bi)(v_1, v_2)) = (av_1 - bv_2, -av_2 - bv_1) \) and \( \overline{\lambda}(v_1, v_2) = (a - bi)(v_1, -v_2) = (av_1 - bv_2, -av_2 - bv_1) \). Also, \( \mathbf{c}((v_1, v_2) + (v_1', v_2')) = \mathbf{c}(v_1 + v_1', v_2 + v_2') = (v_1 + v_1', -v_2 - v_2') = \mathbf{c}(v_1, v_2) + \mathbf{c}(v_1', v_2') \) and, for \( r \) real, \( r(v_1, v_2) = (rv_1, rv_2) \) so \( \mathbf{c}(r(v_1, v_2)) = (rv_1, -rv_2) \). Hence \( \mathbf{c} \) is real-linear.

Thus we've checked the hint.

The second statement that we are to prove, that \( (v_1, v_2) \in \text{null}(\mathbf{S} - \lambda I)^k \) only if \( (v_1, -v_2) \in \text{null}(\mathbf{S} - \overline{\lambda} I)^k \) contains the first statement \( \lambda \) an eigenvalue only if \( \overline{\lambda} \) is in the case \( k = 1 \), so we only prove the more general claim. Suppose \( \mathbf{S}(\mathbf{S} - \lambda I)^k(v_1, v_2) = (0,0) \). Then \( (0,0) = \mathbf{c}((\mathbf{S} - \lambda I)^k(v_1, v_2)) \). We show that \( \mathbf{c}((\mathbf{S} - \lambda I)^k(v_1, v_2)) = (\mathbf{S} - \overline{\lambda} I)^k \mathbf{c}(v_1, v_2) \), which suffices to prove the result since then \( (\mathbf{S} - \overline{\lambda} I)^k(v_1, -v_2) = (0,0) \).

The proof is by induction. In the base case \( k = 0 \) it is certainly true that \( \mathbf{c}((\mathbf{S} - \lambda I)^0(v_1, v_2)) = \mathbf{c}(v_1, v_2) = (\mathbf{S} - \overline{\lambda} I)^0 \mathbf{c}(v_1, v_2) \). Moreover, we know from the hint that \( \mathbf{c}(\mathbf{S}(v_1, v_2)) = \mathbf{S}(\mathbf{c}(v_1, v_2)) \) and \( \mathbf{c}(\lambda \mathbf{I}(v_1, v_2)) = \overline{\lambda} \mathbf{I} \mathbf{c}(v_1, v_2) \) so that, for all \( (v_1, v_2) \in W \), \( \mathbf{c}((\mathbf{T} - \lambda I)(v_1, v_2)) = (\mathbf{T} - \overline{\lambda} I) \mathbf{c}(v_1, v_2) \) (we've used the real-linearity of \( \mathbf{c} \)). Assuming inductively that \( \mathbf{c}((\mathbf{T} - \lambda I)^k(v_1, v_2)) = (\mathbf{T} - \overline{\lambda} I)^k \mathbf{c}(v_1, v_2) \), we have that \( \mathbf{c}((\mathbf{T} - \lambda I)^{k+1}(v_1, v_2)) = (\mathbf{T} - \overline{\lambda} I) \mathbf{c}((\mathbf{T} - \lambda I)^k(v_1, v_2)) = (\mathbf{T} - \overline{\lambda} I)(\mathbf{T} - \overline{\lambda} I)^k \mathbf{c}(v_1, v_2) = (\mathbf{T} - \overline{\lambda} I)^{k+1} \mathbf{c}(v_1, v_2) \). Thus our claim holds by induction, which completes the proposition.