# MATH 113: PROBLEM SET 9

Do Axler, p.125:24,26,27,28,30, p.158:4,11,14,17, p.245:10,16 (for problems involving traces, you may assume that $V$ is complex), and the following problem.

**Problem 1.** Let $V$ be a vector space over $\mathbb{F}$. Recall that $V^* = \mathcal{L}(V, \mathbb{F})$ is the dual of $V$, and if $V$ is finite dimensional and $(v_1, \ldots, v_m)$ is a basis of $V$, the *dual basis* $(f_1, \ldots, f_m)$ of $V^*$ is given by $f_i(v_r) = 0$ if $i \neq r$, $f_i(v_r) = 1$ if $i = r$.

(1) If $v \in V$, show that $\iota(v) : V^* \to \mathbb{F}$ defined by $\iota(v)(f) = f(v)$, $f \in V^*$, is in fact a linear map, i.e. $\iota(v) \in \mathcal{L}(V^*, \mathbb{F}) = (V^*)^*$.

(2) Show that $\iota : V \to (V^*)^*$ is a linear map, i.e. $\iota(v_1 + v_2) = \iota(v_1) + \iota(v_2)$ and $\iota(cv) = c\iota(v)$, $v, v_1, v_2 \in V$, $c \in \mathbb{F}$.

(3) Suppose now that $V$ is finite dimensional. Show that $\iota : V \to (V^*)^*$ is an isomorphism. (Note that $\iota$ does *not* depend on any choices, such as bases.)

(4) If $V$ is a finite dimensional inner product space with inner product $\langle ., . \rangle$ and with basis $(v_1, \ldots, v_m)$, another basis of $V^*$ is given by $(e_1, \ldots, e_m)$ where $e_i(v) = \langle v, v_i \rangle$ (thus, $e_i$ is the image of $v_i$ under the identification of $V$ with $V^*$ given by the inner product). Show that $(e_1, \ldots, e_m) = (f_1, \ldots, f_m)$ if and only if $(v_1, \ldots, v_m)$ is an orthonormal basis of $V$.