MATH 113: PRACTICE MIDTERM

Each problem is 20 points. Attempt all problems.

This is a closed book, closed notes exam, with no calculators allowed (they shouldn't be useful anyway). You may use any theorem, proposition, etc., proved in class or in the book *provided that you quote it precisely.* Make sure that you justify your answer to each question, including the verification that all assumptions of any theorem you quote hold. Try to be brief though.

If on a problem you cannot do part (1) or (2), you may assume its result for the subsequent parts.

Allotted time: 60 minutes.

Problem 1. Let $\mathcal{P}^m(\mathbb{R})$ denote the set of polynomials of degree $\leq m$ on $\mathbb{R}$ with real coefficients. Suppose $(p_0, p_1, \ldots, p_m)$ is a list of elements of $\mathcal{P}^m(\mathbb{R})$ satisfying $\int_0^1 p_j(x)\,dx = 0$, $j = 0, 1, \ldots, m$. Show that $(p_0, p_1, \ldots, p_m)$ is linearly dependent.

Solution 1. Note that $\dim \mathcal{P}^m(\mathbb{R}) = m+1$ as proved in the book. If $(p_0, p_1, \ldots, p_m)$ is linearly independent, it is a basis since this list has length $m+1$. So every $p \in \mathcal{P}^m(\mathbb{R})$ can be written as $p = \sum_{j=0}^m a_j p_j$. In particular,

$$\int_0^1 p(x)\,dx = \sum_{j=0}^m a_j \int_0^1 p_j(x)\,dx = 0$$

for every $p \in \mathcal{P}^m(\mathbb{R})$.

But the polynomial $p = 1$ is in $\mathcal{P}^m(\mathbb{R})$, and $\int_0^1 1\,dx = 1 \neq 0$. This contradicts the result of the previous paragraph, so $(p_0, \ldots, p_m)$ is linearly dependent.

Problem 2. Consider the linear map

$$T : \mathbb{F}^4 \to \mathbb{F}^2,\ T(x_1, x_2, x_3, x_4) = (x_1 + x_3, x_1 - 2x_2 + x_4).$$

(1) Show that T is surjective.
(2) What is $\dim \operatorname{null} T$?
(3) Find a basis for $\operatorname{null} T$.

Solution 2. For $(y_1, y_2) \in \mathbb{F}^2$, note that $T(0, 0, y_1, y_2) = (y_1, y_2)$, so T is indeed surjective. Note that thus $\dim \operatorname{range} T = \dim \mathbb{F}^2 = 2$.

As $4 = \dim \mathbb{F}^4 = \dim \operatorname{range} T + \dim \operatorname{null} T = 2 + \dim \operatorname{null} T$, $\dim \operatorname{null} T = 2$.

$(x_1, x_2, x_3, x_4) \in \operatorname{null} T$ means that $x_1 + x_3 = 0$ and $x_1 - 2x_2 + x_4 = 0$, so $x_3 = -x_1$ and $x_4 = -x_1 + 2x_2$. Correspondingly, every element of $\operatorname{null} T$ is of the form

$$(x_1, x_2, -x_1, -x_1 + 2x_2) = x_1(1, 0, -1, -1) + x_2(0, 1, 0, 2),$$

so $((1, 0, -1, -1), (0, 1, 0, 2))$ spans $\operatorname{null} T$. As the latter is 2-dimensional, every spanning list of length 2 is a basis, in particular this one is.

Problem 3. If V is a vector space, $V^* = \mathcal{L}(V, \mathbb{F})$ is called its (algebraic) dual. If V is finite dimensional, and $(v_1, \ldots, v_n)$ is a basis for V, define $e_j \in V^*$ by

$$e_j(\sum a_k v_k) = a_j.$$

Show that $(e_1, \ldots, e_n)$ is a basis for V^* (called the basis dual to $(v_1, \ldots, v_n)$).

(Hint: to show that $(e_1, \ldots, e_n)$ spans, given $f \in V^*$, you want to find c_j such that $f = \sum c_j e_j$. To find these, calculate $f(v_k)$, and then show that with this choice $f = \sum c_j e_j$ indeed.)

Solution 3. First, note that $e_j(v_k) = 1$ if $j = k$, and is equal to 0 otherwise.

We now show that the e_j are linearly independent in V^*. Indeed, if $\sum_{j=1}^n c_j e_j = 0$ for some $c_j \in \mathbb{F}$, then for all k, $0 = \sum_{j=1}^n c_j e_j(v_k) = c_k$, proving linear independence.

On the other hand, to show spanning, if $f \in V^*$, we need to find $c_j \in \mathbb{F}$ such that $f = \sum_j c_j e_j$. For such c_j we would have $f(v_k) = \sum_j c_j e_j(v_k) = c_k$. So with this motivation we *define* $c_j = f(v_j)$, and note that

$$(\sum_j f(v_j)e_j)(v_k) = \sum_j f(v_j)e_j(v_k) = f(v_k),$$

so the linear maps f and $\sum_j f(v_j)e_j$ agree on the basis $(v_1, \ldots, v_n)$ of V, hence on all of V, i.e. they are equal. Therefore $(e_1, \ldots, e_n)$ spans V^*.

An alternative way of proceeding is to recall that $\dim \mathcal{L}(V, \mathbb{F}) = (\dim V)(\dim \mathbb{F}) = \dim V$, so by the linear independence of $(e_1, \ldots, e_n)$, it is a basis.

Problem 4. Suppose that V, W are finite dimensional vector spaces, and $\dim W < \dim V$. Suppose that $T \in \mathcal{L}(V, W)$, $S \in \mathcal{L}(W, V)$. Show that $ST \in \mathcal{L}(V, V)$ is not injective and is not surjective.

Solution 4. If T is injective then $\dim W \geq \dim V$, as proved in class (and in the book!), which is not the case, so T is not injective. So there is some $u \neq 0$ in $\operatorname{null} T$. But for $v \in \operatorname{null} T$ we have $STv = S0 = 0$, so $v \in \operatorname{null} ST$, i.e. $\operatorname{null} T \subset \operatorname{null}(ST)$. Therefore $\operatorname{null}(ST)$ has a nonzero element u, so ST is not injective.

If S is surjective then $\dim V \leq \dim W$, which is not the case, so S is not surjective. Thus, there is $v \in V$ such that $v \neq Sw$ for any $w \in W$. But if ST is surjective, then there is $u \in V$ such that $v = (ST)u$, so $v = S(Tu)$, and $Tu \in W$. This contradicts the preceeding statement, so ST is not surjective.

Problem 5. Suppose that $(v_1, \ldots, v_n)$ is a list of vectors in a vector space V, and c_{ij}, $i > j$, are given elements of $\mathbb{F}$. Let

$$w_i = v_i + \sum_{j:j<i} c_{ij} v_j.$$

(1) Show that if $(v_1, \ldots, v_n)$ is linearly independent, then so is $(w_1, \ldots, w_n)$. (Hint: try to find a pattern by looking at the special cases $n = 2$ and $n = 3$ at first.)
(2) Prove that if $(v_1, \ldots, v_n)$ is a basis of V then so is $(w_1, \ldots, w_n)$.

Solution 5. Part (2) follows from part (1) immediately, as if $(v_1, \ldots, v_n)$ is a basis, then any list of length n which is linearly independent is a basis, hence this is true for $(w_1, \ldots, w_n)$. So it suffices to prove (1).

So suppose that $(v_1, \ldots, v_n)$ is linearly independent and the w_i are defined as above. Note that $w_1 = v_1 \neq 0$ as $(v_1, \ldots, v_n)$ is a basis. We have seen that if $(w_1, \ldots, w_n)$ is linearly dependent (and $w_1 \neq 0$, which we know), then there is some $k \geq 2$ and some constants $a_\ell \in \mathbb{F}$, $\ell = 1, \ldots, k-1$, such that $w_k = \sum_{\ell=1}^{k-1} a_\ell w_\ell$. But $w_\ell \in \operatorname{span}(v_1, \ldots, v_\ell)$ by construction, so the right hand side is in

$\text{span}(v_1, \ldots, v_{k-1})$, hence so is the left hand side, i.e. w_k. As $w_k = v_k + \sum_{j:j<k} c_{kj} v_j$ and $\sum_{j:j<k} c_{kj} v_j \in \text{span}(v_1, \ldots, v_{k-1})$, we deduce that $v_k = w_k - \sum_{j:j<k} c_{kj} v_j \in \text{span}(v_1, \ldots, v_{k-1})$, contradicting that $(v_1, \ldots, v_n)$ is linearly independent. Thus, $(w_1, \ldots, w_n)$ is linearly independent, finishing the proof.