MATH 113: PRACTICE MIDTERM

Each problem is 20 points. Attempt all problems.

This is a closed book, closed notes exam, with no calculators allowed (they shouldn't be useful anyway). You may use any theorem, proposition, etc., proved in class or in the book *provided that you quote it precisely.* Make sure that you justify your answer to each question, including the verification that all assumptions of any theorem you quote hold. Try to be brief though.

If on a problem you cannot do part (1) or (2), you may assume its result for the subsequent parts.

Allotted time: 60 minutes.

Problem 1. Let $\mathcal{P}^m(\mathbb{R})$ denote the set of polynomials of degree $\leq m$ on $\mathbb{R}$ with real coefficients. Suppose $(p_0, p_1, \ldots, p_m)$ is a list of elements of $\mathcal{P}^m(\mathbb{R})$ satisfying $\int_0^1 p_j(x)\,dx = 0$, $j = 0, 1, \ldots, m$. Show that $(p_0, p_1, \ldots, p_m)$ is linearly dependent.

Problem 2. Consider the linear map

$$T : \mathbb{F}^4 \to \mathbb{F}^2,\ T(x_1, x_2, x_3, x_4) = (x_1 + x_3, x_1 - 2x_2 + x_4).$$

(1) Show that T is surjective.
(2) What is $\dim \operatorname{null} T$?
(3) Find a basis for $\operatorname{null} T$.

Problem 3. If V is a vector space, $V^* = \mathcal{L}(V, \mathbb{F})$ is called its (algebraic) dual. If V is finite dimensional, and $(v_1, \ldots, v_n)$ is a basis for V, define $e_j \in V^*$ by

$$e_j(\sum a_k v_k) = a_j.$$

Show that $(e_1, \ldots, e_n)$ is a basis for V^* (called the basis dual to $(v_1, \ldots, v_n)$).

(Hint: to show that $(e_1, \ldots, e_n)$ spans, given $f \in V^*$, you want to find c_j such that $f = \sum c_j e_j$. To find these, calculate $f(v_k)$, and then show that with this choice $f = \sum c_j e_j$ indeed.)

Problem 4. Suppose that V, W are finite dimensional vector spaces, and $\dim W < \dim V$. Suppose that $T \in \mathcal{L}(V, W)$, $S \in \mathcal{L}(W, V)$. Show that $ST \in \mathcal{L}(V, V)$ is not injective and is not surjective.

Problem 5. Suppose that $(v_1, \ldots, v_n)$ is a list of vectors in a vector space V, and c_{ij}, $i > j$, are given elements of $\mathbb{F}$. Let

$$w_i = v_i + \sum_{j:j<i} c_{ij} v_j.$$

(1) Show that if $(v_1, \ldots, v_n)$ is linearly independent, then so is $(w_1, \ldots, w_n)$. (Hint: try to find a pattern by looking at the special cases $n = 2$ and $n = 3$ at first.)
(2) Prove that if $(v_1, \ldots, v_n)$ is a basis of V then so is $(w_1, \ldots, w_n)$.