# MATH 113: MIDTERM

Each problem is 20 points. Attempt all problems. Problem 5, part (5) is extra credit only, so work on it *only if* you have completed all other problems.

This is a closed book, closed notes exam, with no calculators allowed (they shouldn't be useful anyway). You may use any theorem, proposition, etc., proved in class or in the book *provided that you quote it precisely.* Make sure that you justify your answer to each question, including the verification that all assumptions of any theorem you quote hold. Try to be brief though.

If on a problem you cannot do part (1), (2) or (3), you may assume its result for the subsequent parts.

Allotted time: 60 minutes.

**Problem 1.** Let $\mathcal{P}^m(\mathbb{R})$ denote the set of polynomials of degree $\leq m$ on $\mathbb{R}$ with real coefficients. Suppose that $m \geq 1$ and $(p_0, p_1, \ldots, p_m)$ is a list of elements of $\mathcal{P}^m(\mathbb{R})$ satisfying $p_j(0) = p_j(2)$, $j = 0, 1, \ldots, m$. Show that $(p_0, p_1, \ldots, p_m)$ is linearly dependent. Is this true for $m = 0$?

**Problem 2.** Show that a vector space $V$ is finite dimensional if and only if there exists a natural number $n$ such that every linearly independent list $(v_1, \ldots, v_k)$, $(v_j \in V,\ j = 1, \ldots, k)$ has length $k \leq n$.

**Problem 3.** Consider the linear map

$$T : \mathbb{F}^4 \to \mathbb{F}^2,\ T(x_1, x_2, x_3, x_4) = (x_4 + 2x_3, 3x_1 - x_2).$$

(1) Show that $T$ is surjective.
(2) What is $\dim \operatorname{null} T$?
(3) Find a basis for $\operatorname{null} T$.

**Problem 4.** Suppose that $V, W$ are finite dimensional vector spaces, $U$ is a subspace of $V$ and $Z$ is a subspace of $W$. Show that there is a linear map $S \in \mathcal{L}(V, W)$ such that $\operatorname{null} S = U$ and $\operatorname{range} S = Z$ if and only if $\dim U + \dim Z = \dim V$.

**Problem 5.** Suppose that $V$ is a vector space over $\mathbb{F}$, and $W$ is a subspace of $V$. For $v \in V$, let

$$[v] = \{v' \in V :\ v' - v \in W\} = \{v + w :\ w \in W\} \subset V.$$

(1) Show that for $v_1, v_2 \in V$, we have $[v_1] = [v_2]$ if and only if $v_1 - v_2 \in W$.
(2) Show that $[v]$ is a subspace of $V$ if and only if $v \in W$.
(3) Show that if $[v_1] = [v_1']$ and $[v_2] = [v_2']$ $(v_1, v_1', v_2, v_2' \in V)$ then $[v_1 + v_2] = [v_1' + v_2']$, and if $c \in \mathbb{F}$, $v_1, v_1' \in V$, $[v_1] = [v_1']$ then $[cv_1] = [cv_1']$.
(4) Let $V/W$ denote the set $\{[v] :\ v \in V\}$, and we define addition and multiplication by scalars on $V/W$ by $[v_1] + [v_2] = [v_1 + v_2]$ for $v_1, v_2 \in V$, and for $c \in \mathbb{F}$, $v \in V$, let $c[v] = [cv]$. (These make sense independent of the choice of $v_1$, etc., by the previous part.) List the properties you would need to check to show that $V/W$ is a vector space over $\mathbb{F}$ with these operations, and show that addition is commutative: $[v_1] + [v_2] = [v_2] + [v_1]$ and the following distributive law holds $c([v_1] + [v_2]) = c[v_1] + c[v_2]$.
(5) (Extra credit only!) Show that if $V$ is finite dimensional, then $V/W$ is finite dimensional and $\dim(V/W) = \dim V - \dim W$. (You may assume that $V/W$ is a vector space.)