MATH 113: MIDTERM

Each problem is 20 points. Attempt all problems. Problem 5, part (5) is extra credit only, so work on it only if you have completed all other problems.

This is a closed book, closed notes exam, with no calculators allowed (they shouldn’t be useful anyway). You may use any theorem, proposition, etc., proved in class or in the book provided that you quote it precisely. Make sure that you justify your answer to each question, including the verification that all assumptions of any theorem you quote hold. Try to be brief though.

If on a problem you cannot do part (1), (2) or (3), you may assume its result for the subsequent parts.

Allotted time: 60 minutes.

Problem 1. Let \( P^m(\mathbb{R}) \) denote the set of polynomials of degree \( \leq m \) on \( \mathbb{R} \) with real coefficients. Suppose that \( m \geq 1 \) and \( (p_0, p_1, \ldots, p_m) \) is a list of elements of \( P^m(\mathbb{R}) \) satisfying \( p_j(0) = p_j(2) \), \( j = 0, 1, \ldots, m \). Show that \( (p_0, p_1, \ldots, p_m) \) is linearly dependent. Is this true for \( m = 0 \)?

Problem 2. Show that a vector space \( V \) is finite dimensional if and only if there exists a natural number \( n \) such that every linearly independent list \( (v_1, \ldots, v_k), (v_j \in V, j = 1, \ldots, k) \) has length \( k \leq n \).

Problem 3. Consider the linear map  

\[ T: \mathbb{F}^4 \rightarrow \mathbb{F}^2, \ T(x_1, x_2, x_3, x_4) = (x_4 + 2x_3, 3x_1 - x_2). \]

(1) Show that \( T \) is surjective.
(2) What is \( \dim \ker T \)?
(3) Find a basis for \( \ker T \).

Problem 4. Suppose that \( V, W \) are finite dimensional vector spaces, \( U \) is a subspace of \( V \) and \( Z \) is a subspace of \( W \). Show that there is a linear map \( S \in \mathcal{L}(V, W) \) such that \( \ker S = U \) and \( \text{range} S = Z \) if and only if \( \dim U + \dim Z = \dim V \).

Problem 5. Suppose that \( V \) is a vector space over \( \mathbb{F} \), and \( W \) is a subspace of \( V \). For \( v \in V \), let  

\[ [v] = \{ v' \in V : v' - v \in W \} = \{ v + w : w \in W \} \subset V. \]

(1) Show that for \( v_1, v_2 \in V \), we have \([v_1] = [v_2]\) if and only if \( v_1 - v_2 \in W \).
(2) Show that \([v]\) is a subspace of \( V \) if and only if \( v \in W \).
(3) Show that if \([v_1] = [v'_1]\) and \([v_2] = [v'_2]\), \((v_1, v'_1, v_2, v'_2 \in V)\) then \([v_1 + v_2] = [v'_1 + v'_2]\), and if \( c \in \mathbb{F} \), \( v_1, v'_1 \in V \), \([v_1] = [v'_1]\) then \([cv_1] = [cv'_1]\).
(4) Let \( V/W \) denote the set \( \{ v : v \in V \} \), and we define addition and multiplication by scalars on \( V/W \) by \([v_1] + [v_2] = [v_1 + v_2]\) for \( v_1, v_2 \in V \), and for \( c \in \mathbb{F} \), \( v \in V \), let \([cv] = [cv]\). (These make sense independent of the choice of \( v_1 \), etc., by the previous part.) List the properties you would need to check to show that \( V/W \) is a vector space over \( \mathbb{F} \) with these operations, and show that addition is commutative: \([v_1] + [v_2] = [v_2] + [v_1]\) and the following distributive law holds \( c([v_1] + [v_2]) = c[v_1] + c[v_2]\). (5) (Extra credit only!) Show that if \( V \) is finite dimensional, then \( V/W \) is finite dimensional and \( \dim(V/W) = \dim V - \dim W \). (You may assume that \( V/W \) is a vector space.)