Math 51: Rotations in Higher Dimensions

One day in section, Kristen asked the interesting question of what a "rotation" in \( \mathbb{R}^{10} \) looks like. This is beyond the course, you’ll never be asked to apply geometric concepts to high dimensions, but if you wonder where all our definitions come from, or want a glimpse into how research mathematicians think, read on. Also, the exercises are good practice for writing proofs.

**Answer 1.** In \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \), every rotation is the composition of two reflections, and a reflection sends a vector \( \mathbf{v} \) to \(-\mathbf{v}\), and fixes \( \{x | x \cdot \mathbf{v} = 0\} \). Except for the words "rotation" and "reflection", everything in that last sentence is algebraic, they make sense in every dimension. So, we can define a reflection to be a linear transformation \( R_{\mathbf{v}} \) such that \( R_{\mathbf{v}}(\mathbf{v}) = -\mathbf{v} \), and \( R_{\mathbf{v}}(x) = x \), for all \( x \) with \( x \cdot \mathbf{v} = 0 \).

**Exercise.** Show that the information above is enough to determine \( R_{\mathbf{v}}(w) \) for all \( w \). (Hint: write \( w = \frac{w \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} + x \) for some \( x \))

We can then define a rotation to be the composition of two reflections, but a little thinking shows this isn’t such a good idea. Here’s why: in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \), it turns out that the composition of four reflections can be rewritten as the composition of two reflections, but that’s not necessarily true in \( \mathbb{R}^n \) for larger \( n \). Since \( R_{\mathbf{v}} \) fixes \( \{x | x \cdot \mathbf{v} = 0\} \), \( R_{\mathbf{v}} R_{\mathbf{u}} \) will fix \( \{x | x \cdot \mathbf{v} = 0, x \cdot \mathbf{u} = 0\} \), and most of the time, this subspace has dimension \( n - 2 \). But, by the same logic, a composition of four reflections might fix a subspace of dimension only \( n - 4 \). (This isn’t quite a proof, but you should feel convinced.) And it would be weird if the composition of two rotations wasn’t a rotation. So a better definition would be that a rotation is the composition of an even number of reflections.

**Answer 2.** What makes rotations so interesting that we want to study them in \( \mathbb{R}^{10} \)? One answer is that, in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \), the rotations are precisely the set of linear transformations that preserve lengths of vectors, angles between vectors and orientation. Lengths and angles come from the dot product, so it seems like a good idea to require that rotations in higher dimensions preserve the dot product (that is, \( T_{\mathbf{u}} T_{\mathbf{v}} = \mathbf{u} \cdot \mathbf{v} \)). A linear transformation preserving the dot product is called an isometry.
Changes in orientation are measured algebraically by the determinant: rotations in $\mathbb{R}^2$ and $\mathbb{R}^3$ have positive determinant, but reflections, which reverse orientation, have negative determinant. So we can define rotations in higher dimensions as the set of isometries with positive determinant.

**Exercise.** Show that, with this definition, the composition of two rotations is a rotation.

It is a nonobvious theorem that these two definitions are in fact the same, that is, that given any isometry with positive determinant, I can write it as a composition of an even number of reflections.

**Exercise.** Show the other direction, i.e., if $T$ is a composition of an even number of reflections, then $T$ is an isometry and $\det T > 0$.

The definition of a reflection that I gave above is used all over mathematics. I don’t think there is a universal definition of rotation in higher dimensions, but I think “the isometries with positive determinant” would be most mathematicians’ answer. As my fellow TA Frederick points out, this set of isometries is called $SO(n)$, and comes up all over mathematics.

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