Let $B$ be a symmetric matrix such that
\[
B \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad B \begin{pmatrix} 5 \\ 7 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 5 \\ 7 \\ 0 \end{pmatrix}
\]
and $\det B = 7$. Find the eigenvalues of $B$ and a basis of $\mathbb{R}^3$ consisting of eigenvectors of $B$.

We are given that
- $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector of $B$ with eigenvalue 1
- $\begin{pmatrix} 5 \\ 7 \\ 0 \end{pmatrix}$ is an eigenvector of $B$ with eigenvalue 3

We also know that $B$ is symmetric, so it can be diagonalised. Hence its determinant is the product of its eigenvalues. So $7 = \det B = 1 \cdot 3 \cdot \lambda$, where $\lambda$ is an eigenvalue of $B$. This gives $\lambda = \frac{7}{3}$. So the eigenvalues of $B$ are $1, 3, \frac{7}{3}$.

[Here is a proof that “the determinant of a diagonalisable matrix is the product of its eigenvalues”: if $A$ is diagonalisable, then $A$ is similar to a diagonal matrix $D$. The diagonal entries of $D$ are the eigenvalues of $A$, possibly with repetition. Similar matrices have the same determinant, so $\det A = \det D$, and the determinant of a diagonal matrix is the product of its diagonal entries. Hence $\det A$ is the product of the eigenvalues of $A$, possibly with repetition.]
“Product of its eigenvalues” is a shorthand phrase; you should think of the determinant of a diagonalisable $n$-by-$n$ matrix as “the product of $n$ numbers, all of which are eigenvalues, and each eigenvalue appears at least once amongst these $n$ numbers”. It is important here that the matrix be diagonalisable, although it doesn’t need to be symmetric.]

Let $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ be an eigenvector of $B$ with eigenvalue $\frac{7}{3}$. $B$ is symmetric, so by spectral theorem, eigenvectors corresponding to distinct eigenvalues are orthogonal. Hence

$$v \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0 \text{ and } v \cdot \begin{bmatrix} 5 \\ 7 \\ 0 \end{bmatrix} = 0$$

which means $v_3 = 0$ and $5v_1 + 7v_2 = 0$. This forces

$$v = c \begin{bmatrix} -7 \\ 5 \\ 0 \end{bmatrix}$$

for some $c$, and $c \neq 0$ since $v \neq 0$. So

$$\begin{bmatrix} -7 \\ 5 \\ 0 \end{bmatrix} = \frac{v}{c}$$

As any nonzero multiple of an eigenvector is also an eigenvector, $\begin{bmatrix} -7 \\ 5 \\ 0 \end{bmatrix}$ is a $\frac{7}{3}$-eigenvector of $B$.

So $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 5 \\ 0 \end{bmatrix}$ are eigenvectors of $B$ corresponding to distinct eigenvalues, so they are linearly independent. Since any linearly independent set of three vectors in $\mathbb{R}^3$ is a basis, $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 5 \\ 0 \end{bmatrix} \right\}$ is an eigenbasis with respect to $B$. 
[It turns out that, if we did not know $\det B = 7$ (as in the original question),
the same three vectors give an eigenbasis, but the possibility of repeated eigen-
values creates some subtle difficulties with the logic. It has taken me 10 days
to understand this snag, so don’t panic if it’s confusing. I can’t imagine such a
difficult question coming up on the exams, at least not intentionally (I believe the
people who wrote this question did not see these additional complications).

Recall that Levandosky states the spectral theorem as “a symmetric matrix has
an orthonormal eigenbasis”, but hidden in the proof is the stronger statement that
“eigenvectors of a symmetric matrix corresponding to different eigenvalues are
orthogonal”. If you look closely, the induction proves even more than this, it shows
that “any orthogonal set of eigenvectors of a symmetric matrix can be extended
to an orthogonal eigenbasis” and using this version of the spectral theorem allows
the above computation to go through.

When I first attempted this question, I didn’t see this stronger version of the
spectral theorem, so I analysed the cases where 1 and 3 are repeated eigenvalues
separately. It’s a bit messy, but not too difficult, I can show you if you want.]