QUIZ COMMENTARY OCT. 30

1. Problems and Solutions

Problem 1. *Write down the general solution to the system of differential equations*

$$\vec{x}' = A\vec{x},$$

when

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

Solution:

In class and in section we've written down solutions to problems like this in a couple different ways, all of which are in essence the same. We need to find two linearly independent solutions, $\vec{x}_1(t)$ and $\vec{x}_2(t)$. Then the general solution will be a general linear combination of them

$$c_1\vec{x}_1(t) + c_2\vec{x}_2(t).$$

First we compute the eigenvalues of A. Since

$$\det(A - \lambda I) = (3 - \lambda)^2,$$

we find that $\lambda = 3$ is the only eigenvalue of A. An eigenvector for this eigenvalue is

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus our first solution is

$$\vec{x}_1(t) = e^{3t}\vec{v} = \begin{pmatrix} e^{3t} \\ 0 \end{pmatrix}.$$

Since A does not have two linearly independent eigenvectors (question to the reader: how do we know this?), in our second solution we must use a generalized eigenvector $\vec{w}$. Our chosen $\vec{w}$ should satisfy

$$(A - 3I)\vec{w} = \vec{v}.$$

More than one $\vec{w}$ will work, so just find one- we choose $\vec{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Now, our second solution is

$$\vec{x}_2(t) = te^{3t}\vec{v} + e^{3t}\vec{w} = \begin{pmatrix} te^{3t} \\ e^{3t} \end{pmatrix}.$$

Thus the general solution is

$$c_1\vec{x}_1(t) + c_2\vec{x}_2(t) = \begin{pmatrix} c_1e^{3t} + c_2te^{3t} \\ c_2e^{3t} \end{pmatrix}.$$

Problem 2. *Write down* e^{At} *for*

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}.$$

Solution:

Let $X(t)$ be the fundamental matrix coming from your solution in part one- in other words, the columns of $X(t)$ are your two linearly independent solutions $\vec{x}_1(t)$ and $\vec{x}_2(t)$ to the differential equation

$$\vec{x}' = A\vec{x}.$$

In my case, I got

$$X(t) = \begin{pmatrix} e^{3t} & te^{3t} \\ 0 & e^{3t} \end{pmatrix}.$$

Now, we compute

$$e^{At} = X(t)X(0)^{-1}.$$

Since

$$X(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

the inverse

$$X(0)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

is the identity matrix, too. Thus we get

$$e^{At} = X(t)I = X(t) = \begin{pmatrix} e^{3t} & te^{3t} \\ 0 & e^{3t} \end{pmatrix}.$$

So, in this case, we'd already essentially computed e^{At}. Of course, if you had found a different fundamental matrix $X(t)$, then you would not necessarily have had $X(0)^{-1} = I$; but you still would have gotten e^{At} when you computed $X(t)X(0)^{-1}$.

Commentary:

It's worth thinking a little bit about what's going on here in the second problem. There are lots of choices for the fundamental matrix $X(t)$ coming from solutions in the first problem, since there are lots of choices for the two linearly independent solutions which form the columns of $X(t)$. How are these choices related to one another? Well, suppose $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are your solutions (the columns of your fundamental matrix $X(t)$.) Plug in $t = 0$, so that you get vectors

$$\vec{x}_1(0) = \vec{v}_1, \quad \vec{x}_2(0) = \vec{v}_2.$$

So $X(0)$ is the matrix whose columns are $\vec{v}_1$ and $\vec{v}_2$. You can think of the matrix $X(t)$ is a solution to an 'initial value problem": the matrix $X(t)$ satisfies the differential equation

$$X(t)' = AX(t)$$

and the initial value

$$X(0) = \begin{pmatrix} \vec{v_1} & \vec{v}_2 \end{pmatrix}.$$

Now, e^{At} is a particular example of a fundamental matrix $X(t)$. Which "initial value" problem does it satisfy? Well, at $t = 0$ we have $e^{At} = e^{0A} = I$, so we see that e^{At} is the solution to the differential equation

$$X(t)' = AX(t)$$

with initial value

$$X(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

A choice of a different $X(t)$ will satisfy the same differential equation, but a different initial value ($X(0)$ instead of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.) That's what multiplying by $X(0)^{-1}$ in the formula $e^{At} = X(t)X(0)^{-1}$

does- it moves your initial value back to the identity, which is what distinguishes e^{At} from the other fundamental matrices $X(t)$.