COHERENT SHEAVES AND CATEGORICAL $\mathfrak{sl}_2$ ACTIONS

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Abstract. We introduce the concept of a geometric categorical $\mathfrak{sl}_2$ action and relate it to that of a strong categorical $\mathfrak{sl}_2$ action. The latter is a special kind of 2-representation in the sense of Rouquier. The main result is that a geometric categorical $\mathfrak{sl}_2$ action induces a strong categorical $\mathfrak{sl}_2$ action. This allows one to apply the theory of strong $\mathfrak{sl}_2$ actions to various geometric situations. Our main example is the construction of a geometric categorical $\mathfrak{sl}_2$ action on the derived category of coherent sheaves on cotangent bundles of Grassmannians.

Contents

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1. Introduction

1.1. **Actions of $\mathfrak{sl}_2$ on categories.** An action of $\mathfrak{sl}_2$ on a finite-dimensional vector space V consists of a direct sum decomposition $V = \oplus V(\lambda)$ into weight spaces and linear maps $E(\lambda) : V(\lambda-1) \to V(\lambda+1)$ and $F(\lambda) : V(\lambda+1) \to V(\lambda-1)$. These maps satisfy the condition

$$E(\lambda-1)F(\lambda-1) - F(\lambda+1)E(\lambda+1) = \lambda \mathrm{I}_{V(\lambda)}. \quad (1)$$

Such an action automatically integrates to the group SL_2. In particular, the reflection element $t = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] \in SL_2$ acts on V, inducing an isomorphism $V(-\lambda) \to V(\lambda)$.

A naïve categorical action of $\mathfrak{sl}_2$ consists of a sequence of categories $\mathcal{D}(\lambda)$ with functors

$$\mathsf{E}(\lambda) : \mathcal{D}(\lambda-1) \to \mathcal{D}(\lambda+1) \text{ and } \mathsf{F}(\lambda) : \mathcal{D}(\lambda+1) \to \mathcal{D}(\lambda-1)$$

between them. These functors should satisfy a categorical version of (1) above,

$$\mathsf{E}(\lambda-1) \circ \mathsf{F}(\lambda-1) \cong \mathrm{I}_{\mathcal{D}(\lambda)}^{\oplus\lambda} \oplus \mathsf{F}(\lambda+1) \circ \mathsf{E}(\lambda+1), \quad \text{for } \lambda \geq 0 \quad (2)$$

and an analogous condition when $\lambda \leq 0$. This is just a naïve notion of categorical $\mathfrak{sl}_2$ action, since ideally there should be morphisms between the functors which induce the isomorphisms (2).

The purpose of this paper and the accompaning papers [CKL1], [CKL2], is to apply categorical $\mathfrak{sl}_2$ actions to the geometric situation where $\mathcal{D}(\lambda)$ is the derived category of coherent sheaves of a variety. The main example discussed in this paper is the case of cotangent bundles to Grassmannians of planes in a fixed N dimensional vector space. Thus we fix N and as k varies we let $\mathcal{D}(N-2k) := DCoh(T^\star(\mathbb{G}(k,N))$.

1.2. **Strong categorical $\mathfrak{sl}_2$ actions and geometric categorical $\mathfrak{sl}_2$ actions.** In this paper we have two main definitions. First we define the notion of a *strong categorical action* of $\mathfrak{sl}_2$ (section 2.1), a modification of definitions due to Chuang-Rouquier [CR], Lauda [L], and Rouquier [R]. These axioms include the additional data of morphisms of functors $X : \mathsf{E} \to \mathsf{E}$ and $T : \mathsf{E}^2 \to \mathsf{E}^2$ which rigidify the isomorphisms (2). In a companion paper, [CKL2], where we prove (using ideas from [CR]) that whenever there is a strong categorical action of $\mathfrak{sl}_2$ whose weight spaces are triangulated categories, then we can construct a triangulated equivalence between $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$.

The second notion introduced in this paper is that of a *geometric categorical* $\mathfrak{sl}_2$ action (section 2.2). This means that we have a sequence of varieties $Y(\lambda)$ and Fourier-Mukai kernels $\mathcal{E}(\lambda), \mathcal{F}(\lambda)$, which are objects in the derived categories of the products $Y(\lambda-1) \times Y(\lambda+1)$. These kernels are required to satisfy the commutation relation (2), but only at the level of cohomology. We also demand that there exist certain deformations $\tilde{Y}(\lambda) \to \mathbb{A}^1$ of $Y(\lambda)$ with some special properties. The idea to impose the existence of deformations was inspired by the work of Huybrechts-Thomas [HT] (see Remark 2.6).

The main theorem of this paper (Theorem 2.7) is that a geometric categorical $\mathfrak{sl}_2$ action gives rise to a strong categorical $\mathfrak{sl}_2$ action when the categories involved are the derived categories of coherent sheaves $D(Y(\lambda))$ and where the functors $\mathsf{E}(\lambda), \mathsf{F}(\lambda)$ are induced by the kernels $\mathcal{E}(\lambda), \mathcal{F}(\lambda)$. Roughly speaking, the morphism $X : \mathcal{E}(\lambda) \to \mathcal{E}(\lambda)[2]$ is the obstruction to deforming $\mathcal{E}(\lambda)$ in the family $\tilde{Y}(\lambda - 1) \times_{\mathbb{A}^1} \tilde{Y}(\lambda+1)$. Sections 4 and 5 are devoted to the proof of this theorem. In practice it is much easier to check that certain geometric constructions give rise to a geometric categorical $\mathfrak{sl}_2$ action rather than a strong categorical $\mathfrak{sl}_2$ action. So Theorem 2.7 provides a bridge between geometry (and the results in [CKL1]) and more formal algebraic/categorical constructions provided by a strong $\mathfrak{sl}_2$ action (such as the equivalences constructed in [CKL2]).

1.3. **Relation to 2-categories of Rouquier and Lauda.** In [R], Rouquier defined a 2-categorical version of quantum $\mathfrak{sl}_2$ (based on work in [CR]). This is closely related (not coincidentally) to our definition of strong categorical $\mathfrak{sl}_2$ action. A strong categorical $\mathfrak{sl}_2$ action immediately gives rise to a 2-functor from Rouquier's 2-category into the 2-category of triangulated categories. Thus another way of viewing Theorem 2.7 is to note that it provides a way of obtaining natural 2-representations of Rouquier's 2-category. A strong categorical $\mathfrak{sl}_2$ action is a slightly more restrictive notion than a 2-representation of Rouquier's 2-category, however, because our definition demands additional conditions on the endomorphism algebras $\mathrm{Ext}(\mathsf{E}(\lambda)^{(r)}, \mathsf{E}(\lambda)^{(r)})$. These additional restrictions are used in the proof of the equivalence of triangulated categories between $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ considered in [CKL2].

In [L] Lauda also constructs a 2-category which categorifies quantum $\mathfrak{sl}_2$. Lauda's definition is similar to Rouquier's but involves some extra technical relations which, from our point of view, are not entirely necessary; by this we mean that the equivalence between $\mathcal{D}(-\lambda)$ and $\mathcal{D}(\lambda)$ constructed in [CKL2] does not require these extra relations. It is not obvious that a geometric categorical $\mathfrak{sl}_2$ action gives rise to a 2-representation of Lauda's 2-category (although it is natural to conjecture that it does). Understanding better the role played by these extra relations in Lauda's definition is an interesting problem.

Rouquier [R] and Khovanov-Lauda [KL] have also defined analogous (and closely related) 2-categories for other Kac-Moody Lie algebras. In a future work [CKL3], we will construct 2-representations of these 2-categories (in the simply-laced case) on derived categories of coherent sheaves on quiver varieties, generalizing the action on cotangent bundles to Grassmannians described below.

1.4. **Contangent bundles to Grassmannians.** Our main example of a geometric categorical $\mathfrak{sl}_2$ action is that of cotangent bundles to Grassmannians. We fix N and consider $Y(N-2k) := T^\star \mathbb{G}(k,N)$ as our varieties. There are natural correspondence between Grassmannians which give us the kernels $\mathcal{E}, \mathcal{F}$. In section 3, we use the results of [CKL1] to prove that this is indeed a geometric categorical $\mathfrak{sl}_2$ action.

Hence as a corollary of this paper and of [CKL2], we obtain an explicit equivalence of triangulated categories between $DCoh(T^\star \mathbb{G}(k,N))$ and $DCoh(T^\star \mathbb{G}(N-k,N))$. This answers an open question raised by papers of Kawamata [K] and Namikawa [Na] (see [CKL2] for more details).

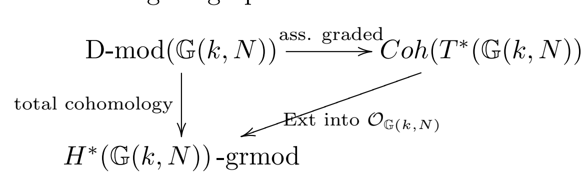
Let us now informally explain how the example of derived categories of coherent sheaves on cotangent bundles to Grassmannians is related to examples to categorical $\mathfrak{sl}_2$ actions studied by Chuang-Rouquier [CR], and Lauda [L].

In section 7.4 of [CR], Chaung-Rouquier defined a strong categorical $\mathfrak{sl}_2$ action (in their sense) on the singular blocks of category $\mathcal{O}$ for $\mathfrak{gl}_N$, following the work of Bernstein-Frenkel-Khovanov [BFK]. By Koszul duality of Beilinson-Ginzburg-Sorgel [BGS], this can be considered as a categorical $\mathfrak{sl}_2$ action on parabolic category $\mathcal{O}$ for $\mathfrak{gl}_N$. We may restrict to the particular parabolic categories considered in [BFK] section 4. Under the Beilinson-Bernstein localization, these categories are equivalent to the categories of B-equivariant D-modules on the Grassmannians $\mathbb{G}(k,N)$. D-modules on $\mathbb{G}(k,N)$ are related to coherent sheaves on $T^*\mathbb{G}(k,N)$ by taking associated graded.

In section 7.7.2 of [CR], Chuang-Rouquier defined a strong categorical $\mathfrak{sl}_2$ action where the categories $\mathcal{D}(N-2k)$ were categories of modules over $H^*(\mathbb{G}(k,N))$. Similarly, in [L], Lauda defined a functor from his 2-category to a 2-category whose 1-morphisms are certain $H^*(\mathbb{G}(k,N)), H^*(\mathbb{G}(l,N))$ graded bimodules. There is a functor from the category of perverse sheaves on $\mathbb{G}(k,N)$ (which is equivalent to D-mod($\mathbb{G}(k,N)$)) to $H^*(\mathbb{G}(k,N))$ graded modules by taking total cohomology.

Also there is a functor from the derived category of coherent sheaves on $T^*(\mathbb{G}(k,N))$ to dg modules over $H^*(\mathbb{G}(k,N))$ by taking $\mathrm{Ext}(\mathcal{O}_{\mathbb{G}(k,N)}, \cdot)$, because $\mathrm{Ext}^*(\mathcal{O}_{\mathbb{G}(k,N)}, \mathcal{O}_{\mathbb{G}(k,N)}) = H^*(\mathbb{G}(k,N))$ (see Remark 5.11 in [CK1]).

To summarize we have the following rough picture:



We expect that these "functors" (if they are made precise as functors) will intertwine the three $\mathfrak{sl}_2$ actions.

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2. Main definitions and results

First, a bit of notational discussion. We will denote composition of functors by juxtaposition and reserve the symbol $\circ$ to denote composition of morphisms. Also we will denote the identity morphism and the identity functor by I. We denote by $\mathbb{G}(k,n)$ the Grassmannian parametrizing k-planes in $\mathbb{C}^n$. We denote by $H^*(\mathbb{G}(k,n))$ the usual cohomology of $\mathbb{G}(k,n)$ but shifted so that it is symmetric with respect to degree zero (equivalently, it is the intersection cohomology). For the purposes of the definition of strong $\mathfrak{sl}_2$ categorification, we will use $\langle\cdot\rangle$ for the grading, whereas later in the paper we will use replace $\langle k\rangle$ by $[k]\{-k\}$. For example,

$$\begin{aligned} H^*(\mathbb{P}^n) &= \mathbb{C}\langle n\rangle \oplus \mathbb{C}\langle n-2\rangle \oplus \cdots \oplus \mathbb{C}\langle -n+2\rangle \oplus \mathbb{C}\langle -n\rangle \\ &= \mathbb{C}[n]\{-n\} \oplus \mathbb{C}[n-2]\{-n+2\} \oplus \cdots \oplus \mathbb{C}[-n+2]\{n-2\} \oplus \mathbb{C}[-n]\{n\} \end{aligned}$$

2.1. **Strong $\mathfrak{sl}_2$ categorification.** Let $\Bbbk$ be a field. A **strong categorical $\mathfrak{sl}_2$ action** consists of the following data.

- A sequence of $\Bbbk$-linear $\mathbb{Z}$-graded, additive categories $\mathcal{D}(-N), \ldots, \mathcal{D}(N)$ which are idempotent complete. Graded means that each category $\mathcal{D}(\lambda)$ has a shift functor $\langle\cdot\rangle$ which is an equivalence.
- Functors
$$\mathsf{E}^{(r)}(\lambda) : \mathcal{D}(\lambda - r) \to \mathcal{D}(\lambda + r) \text{ and } \mathsf{F}^{(r)}(\lambda) : \mathcal{D}(\lambda + r) \to \mathcal{D}(\lambda - r)$$
for $r \geq 0$ and $\lambda \in \mathbb{Z}$. We will usually write $\mathsf{E}(\lambda)$ for $\mathsf{E}^{(1)}(\lambda)$ and $\mathsf{F}(\lambda)$ for $\mathsf{F}^{(1)}(\lambda)$. It is convenient to set $\mathsf{E}^{(0)}(\lambda)$ and $\mathsf{F}^{(0)}(\lambda)$ to be the identity functor I on $Y(\lambda)$.
- Morphisms
$$\eta : \mathrm{I} \to \mathsf{F}^{(r)}(\lambda)\mathsf{E}^{(r)}(\lambda)\langle r\lambda\rangle \text{ and } \eta : \mathrm{I} \to \mathsf{E}^{(r)}(\lambda)\mathsf{F}^{(r)}(\lambda)\langle -r\lambda\rangle$$
$$\varepsilon : \mathsf{F}^{(r)}(\lambda)\mathsf{E}^{(r)}(\lambda) \to \mathrm{I}\langle r\lambda\rangle \text{ and } \varepsilon : \mathsf{E}^{(r)}(\lambda)\mathsf{F}^{(r)}(\lambda) \to \mathrm{I}\langle -r\lambda\rangle.$$
- Morphisms
$$\iota : \mathsf{E}^{(r+1)}(\lambda)\langle r\rangle \to \mathsf{E}(\lambda + r)\mathsf{E}^{(r)}(\lambda - 1) \text{ and } \pi : \mathsf{E}(\lambda + r)\mathsf{E}^{(r)}(\lambda - 1) \to \mathsf{E}^{(r+1)}(\lambda)\langle -r\rangle.$$
- Morphisms
$$X(\lambda) : \mathsf{E}(\lambda)\langle -1\rangle \to \mathsf{E}(\lambda)\langle 1\rangle \text{ and } T(\lambda) : \mathsf{E}(\lambda+1)\mathsf{E}(\lambda-1)\langle 1\rangle \to \mathsf{E}(\lambda+1)\mathsf{E}(\lambda-1)\langle -1\rangle.$$

Let $\mathrm{Hom}(\mathcal{D}(\lambda), \mathcal{D}(\lambda'))$ be the category of additive functors which commute with the shifts. This is also a idempotent complete, additive category.

Remark 2.1. The fact that $\mathcal{D}(\lambda)$ is idempotent complete is useful because then objects have unique (up to isomorphism) direct sum decompositions.

On this data we impose the following additional conditions.

- The morphism η and ε are units and counits of adjunctions
  (i) $\mathsf{E}^{(r)}(\lambda)_R = \mathsf{F}^{(r)}(\lambda)\langle r\lambda\rangle$ for $r \geq 0$

(ii) $\mathsf{E}^{(r)}(\lambda)_L = \mathsf{F}^{(r)}(\lambda)\langle -r\lambda \rangle$ for $r \geq 0$

- Es compose as
$$\mathsf{E}^{(r_2)}(\lambda + r_1)\mathsf{E}^{(r_1)}(\lambda - r_2) \cong \mathsf{E}^{(r_1+r_2)}(\lambda) \otimes_{\Bbbk} H^*(\mathbb{G}(r_1, r_1 + r_2))$$
For example,
$$\mathsf{E}(\lambda + 1)\mathsf{E}(\lambda - 1) \cong \mathsf{E}^{(2)}(\lambda)\langle -1 \rangle \oplus \mathsf{E}^{(2)}(\lambda)\langle 1 \rangle.$$
(By adjointness the Fs compose similarly.) In the case $r_1 = r$ and $r_2 = 1$ we also require that the maps
$$\oplus_{i=0}^r (X(\lambda + r)^i I) \circ \iota\langle -2i \rangle : \mathsf{E}^{(r+1)}(\lambda) \otimes_{\Bbbk} H^\star(\mathbb{P}^r) \to \mathsf{E}(\lambda + r)\mathsf{E}^{(r)}(\lambda - 1)$$
and
$$\oplus_{i=0}^r \pi\langle 2i \rangle \circ (X(\lambda + r)^i I) : \mathsf{E}(\lambda + r)\mathsf{E}^{(r)}(\lambda - 1) \to \mathsf{E}^{(r+1)}(\lambda) \otimes_{\Bbbk} H^\star(\mathbb{P}^r)$$
are isomorphisms. We also have the analogous condition when $r_1 = 1$ and $r_2 = r$.

Remark 2.2. Intuitively, ι maps into the "bottom" factor of
$$\mathsf{E}(\lambda + r)\mathsf{E}^{(r)}(\lambda - 1) \cong \mathsf{E}^{(r+1)}(\lambda) \otimes_{\Bbbk} H^\star(\mathbb{P}^r)$$
while π maps out of the "top" factor.

- If $\lambda \leq 0$ then
$$\mathsf{F}(\lambda + 1)\mathsf{E}(\lambda + 1) \cong \mathsf{E}(\lambda - 1)\mathsf{F}(\lambda - 1) \oplus \mathrm{I} \otimes_{\Bbbk} H^\star(\mathbb{P}^{-\lambda-1}).$$
The isomorphism is induced by
$$\sigma + \sum_{j=0}^{-\lambda-1} (IX(\lambda + 1)^j) \circ \eta : \mathsf{E}(\lambda - 1)\mathsf{F}(\lambda - 1) \oplus \mathrm{I} \otimes_{\Bbbk} H^\star(\mathbb{P}^{-\lambda-1}) \xrightarrow{\sim} \mathsf{F}(\lambda + 1)\mathsf{E}(\lambda + 1)$$
where σ is the composition of maps
$$\begin{aligned} \mathsf{E}(\lambda - 1)\mathsf{F}(\lambda - 1) &\xrightarrow{\eta II} \mathsf{F}(\lambda + 1)\mathsf{E}(\lambda + 1)\mathsf{E}(\lambda - 1)\mathsf{F}(\lambda - 1)\langle \lambda + 1 \rangle \\ &\xrightarrow{IT(\lambda)\mathrm{I}} \mathsf{F}(\lambda + 1)\mathsf{E}(\lambda + 1)\mathsf{E}(\lambda - 1)\mathsf{F}(\lambda - 1)\langle \lambda - 1 \rangle \\ &\xrightarrow{II\epsilon} \mathsf{F}(\lambda + 1)\mathsf{E}(\lambda + 1). \end{aligned}$$
Similarly, if $\lambda \geq 0$ then
$$\mathsf{E}(\lambda - 1)\mathsf{F}(\lambda - 1) \cong \mathsf{F}(\lambda + 1)\mathsf{E}(\lambda + 1) \oplus \mathrm{I} \otimes_{\Bbbk} H^\star(\mathbb{P}^{\lambda-1})$$
with the isomorphism induced in the same way as above.
- The Xs and Ts satisfy the nil affine Hecke relations:
  (i) $T(\lambda)^2 = 0$
  (ii) $(IT(\lambda - 1)) \circ (T(\lambda + 1)I) \circ (IT(\lambda - 1)) = (T(\lambda + 1)I) \circ (IT(\lambda - 1)) \circ (T(\lambda + 1)I)$ as endomorphisms of $\mathsf{E}(\lambda - 2)\mathsf{E}(\lambda)\mathsf{E}(\lambda + 2)$.
  (iii) $(X(\lambda + 1)I) \circ T(\lambda) - T(\lambda) \circ (IX(\lambda - 1)) = I = -(IX(\lambda - 1)) \circ T(\lambda) + T(\lambda) \circ (X(\lambda + 1))$ as endomorphisms of $\mathsf{E}(\lambda - 1)\mathsf{E}(\lambda + 1)$.
- For $r \geq 0$ we have $\mathrm{Hom}(\mathsf{E}^{(r)}(\lambda), \mathsf{E}^{(r)}(\lambda)\langle i \rangle) = 0$ if $i < 0$ while $\mathrm{End}(\mathsf{E}^{(r)}(\lambda)) = \Bbbk \cdot \mathrm{I}$.

Remark 2.3. A strong categorical $\mathfrak{sl}_2$ action is the same thing as a idempotent complete, integrable, graded representation of Rouquier's 2-category in the 2-category of $\Bbbk$-linear categories [R], along with the above extra condition on $\mathrm{Ext}(\mathsf{E}^{(r)}(\lambda), \mathsf{E}^{(r)}(\lambda))$.

Remark 2.4. Although we have categories $\mathcal{D}(\lambda)$ corresponding to weights $-N \le \lambda \le N$ the Es and Fs jump by an even amount from an odd weight to odd weight or from an even weight to an even weight. So we can separate our analysis into studying the odd and even weights. It will therefore often be convenient to assume that $\mathcal{D}(-N+1), \mathcal{D}(-N+3), \dots, \mathcal{D}(N-3), \mathcal{D}(N-1)$ are empty.

2.2. Geometric categorical $\mathfrak{sl}_2$ action.

2.2.1. *A few preliminaries.* If X is a variety we denote by $D(X)$ the bounded derived category of coherent sheaves on X. An object $\mathcal{P} \in D(X \times Y)$ whose support is proper over Y induces a Fourier-Mukai (FM) functor $\Phi_{\mathcal{P}} : D(X) \to D(Y)$ via $(\cdot) \mapsto \pi_{2*}(\pi_1^* \otimes \mathcal{P})$ (where every operation is derived). One says that $\mathcal{P}$ is the FM kernel which induces $\Phi_{\mathcal{P}}$. The right and left adjoints $\Phi_{\mathcal{P}}^R$ and $\Phi_{\mathcal{P}}^L$ are induced by $\mathcal{P}_R := \mathcal{P}^\vee \otimes \pi_2^* \omega_X[\dim(X)]$ and $\mathcal{P}_L := \mathcal{P}^\vee \otimes \pi_1^* \omega_Y[\dim(Y)]$ respectively.

If $\mathcal{Q} \in D(Y \times Z)$ then $\Phi_{\mathcal{Q}} \Phi_{\mathcal{P}} \cong \Phi_{\mathcal{Q} * \mathcal{P}} : D(X) \to D(Y)$ where $\mathcal{Q} * \mathcal{P} = \pi_{13*}(\pi_{12}^* \mathcal{P} \otimes \pi_{23}^* \mathcal{Q})$ is the convolution product (see also [CK1] section 3.1).

If X carries a $\mathbb{C}^\times$ action then we can also consider the bounded derived category of $\mathbb{C}^\times$-equivariant coherent sheaves on X, denoted $D^{\mathbb{C}^\times}(X)$. On X the sheaf $\mathcal{O}_X\{k\}$ denotes the structure sheaf shifted with respect to the $\mathbb{C}^\times$ action so that if $f \in \mathcal{O}_X(U)$ is a local function then viewed as a section $f' \in \mathcal{O}_X\{k\}(U)$ we have $t \cdot f' = t^{-k}(t \cdot f)$. We will denote by $\{k\}$ the operation of tensoring with $\mathcal{O}_X\{k\}$.

2.2.2. *Definition.* Once again we fix a base field $\Bbbk$. A **geometric categorical $\mathfrak{sl}_2$ action** consists of the following data.

- A sequence of smooth varieties $Y(-N), Y(-N+1), \dots, Y(N-1), Y(N)$ over $\Bbbk$ equipped with $\mathbb{C}^\times$-actions and with $\mathbb{C}^\times$-equivariant FM kernels
$$\mathcal{E}^{(r)}(\lambda) \in D(Y(\lambda - r) \times Y(\lambda + r)) \text{ and } \mathcal{F}^{(r)}(\lambda) \in D(Y(\lambda + r) \times Y(\lambda - r)).$$
We will usually write $\mathcal{E}(\lambda)$ for $\mathcal{E}^{(1)}(\lambda)$ and $\mathcal{F}(\lambda)$ for $\mathcal{F}^{(1)}(\lambda)$ while one should think of $\mathcal{E}^{(0)}(\lambda)$ and $\mathcal{F}^{(0)}(\lambda)$ as $\mathcal{O}_\Delta$.
- For each $Y(\lambda)$ a flat deformation $\tilde{Y}(\lambda) \to \mathbb{A}^1_{\Bbbk}$ carrying a $\mathbb{C}^\times$-action which maps fibres to fibres and acts on the base via $x \mapsto t^2 x$ (where $t \in \mathbb{C}^\times$). We call this a compatible $\mathbb{C}^\times$-action.

Remark 2.5. Strictly speaking we only need a first order deformation $\tilde{Y}(\lambda) \to \mathrm{Spec}(\Bbbk[x]/x^2)$ but in all our examples we obtain such a first order deformation from a natural deformation over $\mathbb{A}^1_{\Bbbk}$. However, we could replace $\mathbb{A}^1_{\Bbbk}$ by $\mathrm{Spec}(\Bbbk[x]/x^2)$ in the rest of paper and very little would change.

On this data we impose the following additional conditions.

- All $\mathcal{E}^{(r)}$s and $\mathcal{F}^{(r)}$s are sheaves (i.e. objects supported in one degree).
- (Adjunctions)
  The $\mathcal{E}^{(r)}$s and $\mathcal{F}^{(r)}$s are left and right adjoints of each other up to shift. More precisely
  (i) $\mathcal{E}^{(r)}(\lambda)_R = \mathcal{F}^{(r)}(\lambda)[r\lambda]\{-r\lambda\}$
  (ii) $\mathcal{E}^{(r)}(\lambda)_L = \mathcal{F}^{(r)}(\lambda)[-r\lambda]\{r\lambda\}$.
- (Composition of $\mathcal{E}$'s and $\mathcal{F}$'s) At the level of cohomology of complexes we have
$$\mathcal{H}^*(\mathcal{E}(\lambda + r) * \mathcal{E}^{(r)}(\lambda - 1)) \cong \mathcal{E}^{(r+1)}(\lambda) \otimes_{\Bbbk} H^\star(\mathbb{P}^r)$$
while in the deformed case we have
$$\mathcal{H}^*(j_{23*}\mathcal{E}(\lambda + r) * j_{12*}\mathcal{E}^{(r)}(\lambda - 1)) \cong \mathcal{E}^{(r+1)}(\lambda)[-r]\{r\} \oplus \mathcal{E}^{(r+1)}(\lambda)[r+1]\{-r-2\}$$
where the j's are the inclusions
$$j_{12} : Y(\lambda - 1 - r) \times Y(\lambda - 1 + r) \to Y(\lambda - 1 - r) \times \tilde{Y}(\lambda - 1 + r)$$
$$j_{23} : Y(\lambda - 1 + r) \times Y(\lambda + 1 + r) \to \tilde{Y}(\lambda - 1 + r) \times Y(\lambda + 1 + r).$$

We also have the same relations if we replace all the $\mathcal{E}$'s by $\mathcal{F}$'s.

- (Commutator relation)
  If $\lambda \le 0$ then
$$\mathcal{F}(\lambda+1) * \mathcal{E}(\lambda+1) \cong \mathcal{E}(\lambda-1) * \mathcal{F}(\lambda-1) \oplus \mathcal{P}$$
  and $\mathcal{H}^*(\mathcal{P}) \cong \mathcal{O}_\Delta \otimes_{\Bbbk} H^\star(\mathbb{P}^{-\lambda-1})$. Moreover $\Delta \not\subset \mathrm{supp}(\mathcal{E}(\lambda-1) * \mathcal{F}(\lambda-1))$.
  In the deformed case we have the exact triangle
$$\begin{matrix} \mathcal{E}(\lambda-1) * \mathcal{F}(\lambda-1)[1]\{-2\} \\ \oplus \mathcal{O}_\Delta[-\lambda]\{\lambda-1\} \end{matrix} \to j_{12*}\mathcal{F}(\lambda+1) * j_{12*}\mathcal{E}(\lambda+1) \to \begin{matrix} \mathcal{E}(\lambda-1) * \mathcal{F}(\lambda-1) \\ \oplus \mathcal{O}_\Delta[\lambda+1]\{-\lambda-1\} \end{matrix}$$
  where j_{12} is the inclusion
$$j_{12} : Y(\lambda) \times Y(\lambda+2) \to Y(\lambda) \times \tilde{Y}(\lambda+2).$$
  Similarly, if $\lambda \ge 0$ then
$$\mathcal{E}(\lambda-1) * \mathcal{F}(\lambda-1) \cong \mathcal{F}(\lambda+1) * \mathcal{E}(\lambda+1) \oplus \mathcal{P}'$$
  and $\mathcal{H}^*(\mathcal{P}') \cong \mathcal{O}_\Delta \otimes_{\Bbbk} H^\star(\mathbb{P}^{\lambda-1})$ and $\Delta \not\subset \mathrm{supp}(\mathcal{F}(\lambda+1) * \mathcal{E}(\lambda+1))$. In the deformed case we have the exact triangle
$$\begin{matrix} \mathcal{F}(\lambda+1) * \mathcal{E}(\lambda+1)[1]\{-2\} \\ \oplus \mathcal{O}_\Delta[\lambda]\{-\lambda-1\} \end{matrix} \to j'_{12*}\mathcal{E}(\lambda-1) * j'_{12*}\mathcal{F}(\lambda-1) \to \begin{matrix} \mathcal{F}(\lambda+1) * \mathcal{E}(\lambda+1) \\ \oplus \mathcal{O}_\Delta[-\lambda+1]\{\lambda-1\} \end{matrix}$$
  where j'_{12} is the inclusion
$$j'_{12} : Y(\lambda) \times Y(\lambda-2) \to Y(\lambda) \times \tilde{Y}(\lambda-2).$$
- $\mathrm{End}(\mathcal{E}^{(r)}(\lambda)) = \Bbbk \cdot \mathrm{I}$ for $r \ge 0$.

Remark 2.6. Having the conditions at the level of cohomology may seem strange but it is often much easier to prove that the cohomologies of two objects are the same than to prove that the objects are isomorphic. Moreover, there are natural examples where isomorphisms hold only at the level of cohomology. The moral is that also having deformations (with the properties described above) allows one to lift isomorphisms from the level of cohomology of objects to isomorphisms of objects.

The idea of imposing the existence of a deformation was inspired by the work of Huybrechts-Thomas on $\mathbb{P}^n$ objects. In particular, in Proposition 1.4 of [HT], they show that under certain conditions, the deformation of a $\mathbb{P}^n$ object is spherical. When $\lambda = -N+1$, $\mathcal{E}(\lambda)$ is a relative $\mathbb{P}^n$ object and we impose a "spherical" condition on the deformation.

2.3. The main result.

Theorem 2.7. *Given a geometric categorical $\mathfrak{sl}_2$ action set*
$$\mathcal{D}(\lambda) := D(Y(\lambda)) \text{ and } \mathsf{E}^{(r)}(\lambda) := \Phi_{\mathcal{E}^{(r)}(\lambda)} \text{ and } \mathsf{F}^{(r)}(\lambda) := \Phi_{\mathcal{F}^{(r)}(\lambda)}$$
where the shift in $\mathcal{D}(\lambda)$ is given by $\langle r \rangle = [r]\{-r\}$. Then there exist morphisms $\iota, \pi, \varepsilon, \eta, X, T$ giving a strong categorical $\mathfrak{sl}_2$ action. Moreover, the choice of the X and T is parametrized by
$$V(-1)^{tr} \times V(-2)^{tr} \times \Bbbk^\times \cong V(1)^{tr} \times V(2)^{tr} \times \Bbbk^\times$$
where $V(\lambda)^{tr} \subset \mathrm{Ext}^2(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$ denotes the linear subspace of transient maps defined in section 5.1. The choices of ι, π, ε and η are unique up to $\Bbbk^\times$.

Remark 2.8. One may very well choose to ignore the $\mathbb{C}^\times$-action and nothing in the statement or proof of Theorem 2.7 would change (except that we would have no $\{\cdot\}$ shift and $\langle \cdot \rangle = [\cdot]$). One reason to include the $\mathbb{C}^\times$ is because it occurs naturally in many of the examples we know and provides another grading which will be useful in future work. Also, there are examples (such as the one below) where the condition $\mathrm{End}(\mathcal{E}^{(r)}(\lambda)) = \Bbbk \cdot \mathrm{I}$ fails if one doesn't work $\mathbb{C}^\times$-equivariantly.

3. The main example

Before proceeding with the proof of main Theorem 2.7 we give an example of a geometric categorical $\mathfrak{sl}_2$ action. We work over the base field $\Bbbk = \mathbb{C}$. The spaces involved will be cotangent bundles to Grassmannians. In [CKL1] we gave an example of a geometric categorical $\mathfrak{sl}_2$ action which is essentially a natural compactification of the one here. However, we prefer the one given here since in some ways it is simpler and more fundamental.

3.1. **Spaces and functors.** Fix $N > 0$. For our spaces $Y(\lambda)$ we will take the total cotangent bundle to the Grassmannian $T^\star\mathbb{G}(k,N)$ where $k = (N-\lambda)/2$. The $\mathbb{C}^\times$ will act naturally on the fibres of the bundle. These spaces have a particularly nice geometric description as

$$T^\star\mathbb{G}(k,N) \cong \{(X,V) : X \in \mathrm{End}(\mathbb{C}^N), 0 \subset V \subset \mathbb{C}^N, \dim(V) = k \text{ and } \mathbb{C}^N \xrightarrow{X} V \xrightarrow{X} 0\}$$

where $\mathrm{End}(\mathbb{C}^N)$ denotes the space of complex $N \times N$ matrices (the notation $\mathbb{C}^N \xrightarrow{X} V \xrightarrow{X} 0$ means that $X(\mathbb{C}^n) \subset V$ and that $X(V) = 0$). The action of $t \in \mathbb{C}^\times$ is by $X \mapsto t^2 X$.

Forgetting X corresponds to the projection $T^\star\mathbb{G}(k,N) \to \mathbb{G}(k,N)$ while forgetting V gives a resolution of the variety

$$\{X \in \mathrm{End}(\mathbb{C}^N) : X^2 = 0 \text{ and } \mathrm{rank}(X) \le \min(k, N-k)\}.$$

On $Y(\lambda) = T^\star\mathbb{G}(k,N)$ we have the tautological vector bundle V as well as the quotient $\mathbb{C}^N/V$.

To describe the kernels $\mathcal{E}$ and $\mathcal{F}$ we will need the correspondences

$$W^r(\lambda) \subset T^\star\mathbb{G}(k+r/2,N) \times T^\star\mathbb{G}(k-r/2,N)$$

defined by

$$\begin{aligned} W^r(\lambda) := \{(X,V,V') :& X \in \mathrm{End}(\mathbb{C}^N), \dim(V) = k + \frac{r}{2}, \dim(V') = k - \frac{r}{2}, 0 \subset V' \subset V \subset \mathbb{C}^N \\ & \mathbb{C}^N \xrightarrow{X} V' \text{ and } V \xrightarrow{X} 0\} \end{aligned}$$

(here as before λ and k are related by the equation $k = (N-\lambda)/2$).

There are two natural projections $\pi_1 : (X,V,V') \mapsto (X,V)$ and $\pi_2 : (X,V,V') \mapsto (X,V')$ from $W^r(\lambda)$ to $Y(\lambda - r)$ and $Y(\lambda + r)$ respectively. Together they give us an embedding

$$(\pi_1, \pi_2) : W^r(\lambda) \subset Y(\lambda - r) \times Y(\lambda + r).$$

Notice that we also have a natural $\mathbb{C}^\times$ action on $W^r(\lambda)$ given by $X \mapsto t^2 X$ so that both π_1 and π_2 are $\mathbb{C}^\times$-equivariant.

On $W^r(\lambda)$ we have two natural tautological bundles, namely $V := \pi_1^*(V)$ and $V' := \pi_2^*(V)$ where the prime on the V' indicates that the vector bundle is the pullback of the tautological bundle by the second projection. We also have natural inclusions

$$0 \subset V' \subset V \subset \mathbb{C}^N \cong \mathcal{O}_{W^r(\lambda)}^{\oplus N}.$$

We now define the kernel $\mathcal{E}^{(r)}(\lambda) \in D(Y(\lambda - r) \times Y(\lambda + r))$ by

$$\mathcal{E}^{(r)}(\lambda) := \mathcal{O}_{W^r(\lambda)} \otimes \det(\mathbb{C}^N/V')^{-r} \det(V)^r \{\frac{r(N-\lambda-r)}{2}\}.$$

Similarly, the kernel $\mathcal{F}^{(r)}(\lambda) \in D(Y(\lambda + r) \times Y(\lambda - r))$ is defined by

$$\mathcal{F}^{(r)}(\lambda) := \mathcal{O}_{W^r(\lambda)} \otimes \det(V'/V)^{\lambda} \{\frac{r(N+\lambda-r)}{2}\}.$$

Notice that here $V' = \pi_2^*(V)$ is the pullback from the projection onto $Y(\lambda - r)$ since we now view $Y(\lambda - r)$ as being the second factor rather than the first.

Remark 3.1. Although $W^r(\lambda)$ is not proper the projections onto $Y(\lambda-r)$ and $Y(\lambda+r)$ are proper since the fibres are subvarieties of Grassmannians. Hence $\mathcal{E}^{(r)}(\lambda)$ and $\mathcal{F}^{(r)}(\lambda)$ induce FM transforms $\mathsf{E}^{(r)}(\lambda)$ and $\mathsf{F}^{(r)}(\lambda)$ between $D(Y(\lambda-r))$ and $D(Y(\lambda+r))$.

3.2. **Deformations.** $Y(\lambda) = T^\star\mathbb{G}(k,N)$ has a natural 2-parameter deformation over $\mathbb{A}^2_{\mathbb{C}}$, whose fibre at the point (x,y) is given by

$$\{(X,V) : X \in \mathrm{End}(\mathbb{C}^N), 0 \subset V \subset \mathbb{C}^N, \dim(V) = k \text{ and } X|_V = x\cdot \mathrm{I}, X|_{\mathbb{C}^N/V} = y \cdot \mathrm{I}\}.$$

Notice that the fibre over $(x,y)=(0,0)$ recovers $T^\star\mathbb{G}(k,N)$. This deformation restricted to the diagonal $x=y$ is actually trivial but if we take any other ray in $\mathbb{A}^2_{\mathbb{C}}$ through the origin we get a non-trivial deformation of $T^\star\mathbb{G}(k,N)$. Which ray we choose is not that important but to make our notation match the one in [CKL1] we choose the anti-diagonal $x+y=0$ to obtain the deformation

$$\tilde{Y}(\lambda) = \{(X,V,x) : x \in \mathbb{C}, X \in \mathrm{End}(\mathbb{C}^N), 0 \subset V \subset \mathbb{C}^N, \dim(V) = k \text{ and } X|_V = x\cdot \mathrm{I}, X|_{\mathbb{C}^N/V} = -x \cdot \mathrm{I}\}$$

where $\lambda = N-2k$. The $\mathbb{C}^\times$-action here acts, like before, by $X \mapsto t^2 X$ and by $x \mapsto t^2 x$. Thus we have a compatible $\mathbb{C}^\times$-action.

3.3. **Relation to categorification of skew Howe duality.** In [CKL1] we constructed a geometric categorical $\mathfrak{sl}_2$ action on varieties which compactified the above cotangent bundles. We now explain how that categorification is related to the one above.

In [CKL1] we fixed integers m,N and defined varieties $Y(k,l)$ and functors $\mathcal{E}^{(r)}(k,l)$ where $k+l=N$. However, only the case when $m=N$ is related to the example above. We now recall these varieties and functors when $m=N$.

We define

$$Y(k,l) := \{\mathbb{C}^N \otimes \mathbb{C}[[z]] = L_0 \subset L_1 \subset L_2 \subset \mathbb{C}^N \otimes \mathbb{C}((z)) : zL_i \subset L_{i-1}, \dim(L_1/L_0) = k, \dim(L_2/L_1) = l\}$$

so that $D(Y(k,l))$ will correspond to the weight space of weight $\lambda = l-k$. The $\mathbb{C}^\times$-action on $Y(k,l)$ is induced by $t \cdot z^k = t^{2k} z^k$ acting on $\mathbb{C}((z))$.

Next we define correspondences

$$W^r(k,l) := \{(L_\bullet, L'_\bullet) : L_1 \subset L'_1, L_2 = L'_2\} \subset Y(k,l) \times Y(k+r, l-r)$$

followed by kernels

$$\mathcal{E}^{(r)}(k,l) := \mathcal{O}_{W^r(k,l)} \otimes \det(L_2/L'_1)^{-r} \det(L_1/L_0)^r \{rk\} \in D(Y(k+r,l-r) \times Y(k,l))$$

and

$$\mathcal{F}^{(r)}(k,l) := \mathcal{O}_{W^r(k,l)} \otimes \det(L'_1/L_1)^{l-r-k} \{r(l-r)\} \in D(Y(k,l) \times Y(k+r,l-r))$$

where (abusing notation a little) L_i denotes the tautological bundle on $Y(k,l)$ whose fibre over the point $(L_0 \subset L_1 \subset L_2)$ is L_i. As usual, one can check that everything here is $\mathbb{C}^\times$-equivariant.

Since $z^2 L_2 \subset L_0$ this means that $\mathbb{C}^N \otimes \mathbb{C}[[z]] \subset L_2 \subset \mathbb{C}^N \otimes z^{-2}\mathbb{C}[[z]]$. Hence the N dimensional space L_2/L_0 is a subspace of the $2N$ dimensional vector space $\mathbb{C}^N \otimes z^{-2}\mathbb{C}[[z]]/C[[z]]$. Define the linear projection $P : \mathbb{C}^N \otimes z^{-2}\mathbb{C}[[z]]/C[[z]] \to \mathbb{C}^N$ by $P(v \otimes z^{-1}) = v$ and $P(v \otimes z^{-2}) = 0$. Now consider the open subvariety

$$U(k,l) := \{L_\bullet : P(L_2) = \mathbb{C}^N\} \subset Y(k,l).$$

The following result is due to [MVy, Theorem 5.3] and [Ng, Lemma 2.3.1]

Lemma 3.2. *$U(k,l) \cong T^\star\mathbb{G}(k,N)$ via the isomorphism*

$$(L_0 \subset L_1 \subset L_2) \mapsto (PzP^{-1}, P(L_1/L_0)) = (X,V).$$

Moreover, this isomorphism is $\mathbb{C}^\times$-equivariant with respect to the $\mathbb{C}^\times$-actions on $Y(k,l)$ and $T^\star\mathbb{G}(k,N)$ defined above.

Proof. We give a sketch. The definition of $U(k,l)$ implies that if $L_\bullet \in U(k,l)$, then P gives an isomorphism between L_2 and $\mathbb{C}^N$. So P takes L_1 to a k-dimensional subspace $V \subset \mathbb{C}^N$ and $0 \xleftarrow{z} L_1 \xleftarrow{z} L_2$ induces the map $0 \xleftarrow{X} V \xleftarrow{X} \mathbb{C}^N$ where $X = PzP^{-1}$. The fact that this isomorphism is $\mathbb{C}^\times$-equivariant follows since the $\mathbb{C}^\times$-actions are given by $X \mapsto t^2 X$ and $z \mapsto t^2 z$. □

Now in [CKL1] we also had deformations

$$\tilde{Y}(k,l) := \{L_\bullet : z|_{L_1/L_0} = x \cdot \mathrm{I}, z|_{L_2/L_1} = -x \cdot \mathrm{I}, \dim(L_1/L_0) = k, \dim(L_2/L_1) = l\}.$$

These were also equipped with $\mathbb{C}^\times$-actions induced by $t \cdot z^k = t^{2k} z^k$ and $t \cdot x = t^2 x$. From [MVy, Theorem 5.3], we obtain the following result:

Lemma 3.3. *The embedding $T^\star\mathbb{G}(k,N) \cong U(k,l) \to Y(k,l)$ of Lemma 3.2 extends to an embedding $\tilde{Y}(\lambda) \to \tilde{Y}(k,l)$ which is compatible with the projections to $\mathbb{A}^1_{\mathbb{C}}$ and the $\mathbb{C}^\times$-actions given above.*

3.4. **Obtaining a geometric categorical $\mathfrak{sl}_2$ action.** In [CKL1], we showed that the varieties $Y(k,l)$ gave a geometric categorical $\mathfrak{sl}_2$ action. Using that result, we can prove the following.

Theorem 3.4. *The varieties $Y(\lambda) = T^*\mathbb{G}(k,n)$ along with the deformations $\tilde{Y}(\lambda)$ and kernels $\mathcal{E}^{(r)}(\lambda)$, $\mathcal{F}^{(r)}(\lambda)$, give a geometric categorical $\mathfrak{sl}_2$ action.*

Of course, the idea is to show that the categorical $\mathfrak{sl}_2$ relations for the $Y(k,l)$ varieties induce the same relations for our open subvarieties $T^\star\mathbb{G}(k,N)$.

To do this we begin with the following observation. Denote by $j : Y(\lambda) = T^\star\mathbb{G}(k,N) \to Y(k,l)$ the natural open immersion. Then it is not difficult to see that

$$(j \times j)^{-1} W^r(k,l) = W^r(\lambda) \subset T^\star\mathbb{G}(k,N) \times T^\star\mathbb{G}(k+r,N)$$

where $\lambda = N - 2k - r$. Even better, we have

$$(j \times 1)^{-1} W^r(k,l) = W^r(\lambda) \subset T^\star\mathbb{G}(k,N) \times Y(k+r, l-r)$$

Moreover, since L_1 and L'_1 on $W^r(k,l)$ correpond to V and V' on $W^r(\lambda)$ it is easy to check that

$$(j \times j)^* \mathcal{E}^{(r)}(k,l) \cong \mathcal{E}^{(r)}(\lambda) \text{ and } (j \times j)^* \mathcal{F}^{(r)}(k,l) = \mathcal{F}^{(r)}(\lambda)$$

and that

$$(j \times 1)^* \mathcal{E}^{(r)}(k,l) \cong (1 \times j)_* \mathcal{E}^{(r)}(\lambda) \text{ and } (j \times 1)^* \mathcal{F}^{(r)}(k,l) = (1 \times j)_* \mathcal{F}^{(r)}(\lambda). \tag{3}$$

We can now make use of the following lemma.

Lemma 3.5. *Suppose that Y_1, Y_2, Y_3 are smooth varieties and U_1, U_2, U_3 are open subvarieties. Let $j_a : U_a \to Y_a$ denote the natural open immersion. Let $F_{12} \in D(Y_1 \times Y_2), F_{23} \in D(Y_2 \times Y_3)$ denote objects on the products and let*

$$F'_{12} := (j_1 \times j_2)^*(F_{12}) \in D(U_1 \times U_2) \text{ and } F'_{23} := (j_2 \times j_3)^*(F_{23}) \in D(U_2 \times U_3).$$

Suppose moreover that

$$(j_1 \times 1)^* F_{12} \cong (1 \times j_2)_* F'_{12} \in D(U_1 \times Y_2) \text{ and } (1 \times j_3)^* F_{23} = (j_2 \times 1)_* F'_{23} \in D(Y_2 \times U_3).$$

Then $F'_{23} * F'_{12} \cong (j_1 \times j_3)^*(F_{23} * F_{12})$

Proof. This follows by a direct calculation. We have

$$\begin{aligned}
(j_1 \times j_3)^*(F_{23} * F_{12}) &\cong (j_1 \times j_3)^* \pi_{13*}(\pi_{12}^* F_{12} \otimes \pi_{23}^* F_{23}) \\
&\cong p_{13*}(j_1 \times 1 \times j_3)^*(\pi_{12}^* F_{12} \otimes \pi_{23}^* F_{23}) \\
&\cong p_{13*}\left((j_1 \times 1 \times j_3)^* \pi_{12}^* F_{12} \otimes (j_1 \times 1 \times j_3)^* \pi_{23}^* F_{23}\right) \\
&\cong p_{13*}\left(p_{12}^*(j_1 \times 1)^* F_{12} \otimes p_{23}^*(1 \times j_3)^* F_{23}\right) \\
&\cong p_{13*}\left(p_{12}^*(1 \times j_2)_* F'_{12} \otimes p_{23}^*(j_2 \times 1)_* F'_{23}\right) \\
&\cong p_{13*}\left((1 \times j_2 \times 1)_* {\pi'_{12}}^* F'_{12} \otimes (1 \times j_2 \times 1)_* {\pi'_{23}}^* F'_{23}\right) \\
&\cong p_{13*}(1 \times j_2 \times 1)_* \left({\pi'_{12}}^* F'_{12} \otimes {\pi'_{23}}^* F'_{23}\right) \\
&\cong \pi'_{13*}({\pi'_{12}}^* F'_{12} \otimes {\pi'_{23}}^* F'_{23}) \\
&\cong F'_{23} * F'_{12}
\end{aligned}$$

where p_{ab} is the projection from $U_1 \times Y_2 \times U_3$ onto the a, b factor and π'_{ab} is the projection from $U_1 \times U_2 \times U_3$ onto the a, b factor. To get the 2nd and 6th isomorphisms we used commutativity of pushing and pulling in a flat base change. To get the 7th isomorphism we used that $1 \times j_2 \times 1$ is an open immersion so tensoring commutes with pushforward. □

Proof of Theorem 3.4. We need to check that the $\mathcal{E}(\lambda)$'s and $\mathcal{F}(\lambda)$'s satisfy all the conditions of having a geometric categorical $\mathfrak{sl}_2$ action.

First, the condition on the adjunction between $\mathcal{E}(\lambda)$'s and $\mathcal{F}(\lambda)$'s follows by a direct computation identical to that in [CKL1].

Now, we compute the composition of $\mathcal{E}(\lambda)$'s. We apply Lemma 3.5 with $U_1 := Y(\lambda - 1 - r), U_2 := Y(\lambda - 1 + r), U_3 := Y(\lambda + 1 + r), Y_1 := Y(k + r, l - r), Y_2 := Y(k, l), Y_3 := Y(k - 1, l + 1)$, where $\lambda = N - 2k - r + 1$. We choose $F_{12} := \mathcal{E}^{(r)}(k, l), F_{23} := \mathcal{E}(k - 1, l + 1)$. The main hypothesis of Lemma 3.5 follows from (3). From the conclusion of Lemma 3.5, we deduce that

$$\mathcal{E}(\lambda + r) * \mathcal{E}^{(r)}(\lambda - 1) \cong (j_1 \times j_3)^*(\mathcal{E}(k - 1, l + 1) * \mathcal{E}^{(r)}(k, l)).$$

Applying $\mathcal{H}^*$ to both sides, and using the fact that the underived pullback $(j_1 \times j_3)^*$ is exact, we obtain

$$\begin{aligned}
\mathcal{H}^*(\mathcal{E}(\lambda + r) * \mathcal{E}^{(r)}(\lambda - 1)) &\cong (j_1 \times j_3)^*(\mathcal{H}^*(\mathcal{E}(k - 1, l + 1) * \mathcal{E}^{(r)}(k, l))) \\
&\cong (j_1 \times j_3)^*(\mathcal{E}^{(r+1)}(k - 1, l + 1) \otimes_{\mathbb{C}} H^\star(\mathbb{P}^r)) \\
&\cong \mathcal{E}^{(r+1)}(\lambda) \otimes_{\mathbb{C}} H^\star(\mathbb{P}^r).
\end{aligned}$$

Thus the relation for the composition of the $\mathcal{E}(\lambda)$'s follows from the corresponding relation for the composition of the $\mathcal{E}(k, l)$'s.

Now, we consider the condition in the deformation. Here we apply the Lemma with Y_1, Y_3, U_1, U_3 as above but with $Y_2 = \tilde{Y}(k, l)$ and $U_2 = \tilde{Y}(\lambda - 1 + r)$. By the same reasoning as above we obtain the condition on composition of $\mathcal{E}(k, l)$'s in the deformed case.

The commutator relation between $\mathcal{E}(\lambda)$'s and $\mathcal{F}(\lambda)$'s follows similarly.

The final thing to check is that $\mathrm{End}(\mathcal{E}^{(r)}(\lambda)) = \mathbb{C} \cdot \mathrm{I}$. This follows if we can show that $\mathrm{End}(\mathcal{O}_{W^r(\lambda)}) = \mathbb{C} \cdot \mathrm{I}$. Now if we forget X then we get a projection map from $W^{(r)}(\lambda)$ onto the flag variety $\mathrm{Flag}(k - \frac{r}{2}, k + \frac{r}{2}, N)$. This gives $W^{(r)}(\lambda)$ the structure of a $\mathbb{C}^n$-bundle over the flag (for some n). The $\mathbb{C}^\times$-action fixes the flag and acts linearly on the fibres of the bundle with zero the only fixed point. This means that the only $\mathbb{C}^\times$-equivariant section of $\mathcal{O}_{W^r(\lambda)}$ is the constant section and we are done. Notice that this would not be true if we ignore the $\mathbb{C}^\times$-action. □

Remark 3.6. It follows immediately from Theorem 3.4 that $U_q(\mathfrak{sl}_2)$ acts on the Grothendieck group

$$K(N) := \bigoplus_{\lambda=-N}^{N} K(\mathcal{D}(\lambda))$$

where $K(\mathcal{D}(\lambda))$ is a $\mathbb{C}[q,q^{-1}]$ module with $-q$ acting by $\{1\}$.

The weight space $K(\mathcal{D}(\lambda))$ has dimension $\dim H^*(T^\star(\mathbb{G}(k,n)))) = \binom{N}{k}$ by the argument in Proposition 7.2 of [CK2]. Hence as a $U_q(\mathfrak{sl}_2)$ representation, $K(N)$ is isomorphic to the Nth tensor power of the irreducible 2-dimensional representation.

4. Obtaining formality from deformations $\tilde{Y}(\lambda)$

The rest of the paper is devoted to the proof of Theorem 2.7. We will assume throughout that we have a fixed geometric categorical $\mathbb{C}^\times$-equivariant $\mathfrak{sl}_2$ action.

The most difficult part of the proof (by far) is to construct the $X(\lambda)$'s and $T(\lambda)$'s so that they satisfy the nil affine Hecke relations. So we first check all the other properties and leave the nil affine Hecke relations until the end.

First notice that the $\mathcal{E}$'s (and $\mathcal{F}$'s) are sheaves so we immediately get $\mathrm{Hom}(\mathcal{E}^{(r)}(\lambda), \mathcal{E}^{(r)}(\lambda)[i]) = 0$ if $i < 0$. Also, since $\mathrm{End}(\mathcal{E}^{(r)}(\lambda)) = \Bbbk \cdot \mathrm{I}$ the $\mathcal{E}$'s (and by adjointness the $\mathcal{F}$'s) are simple objects.

4.1. Some deformation theory.

We begin with some general deformation theory. Our deformations will be over $\mathbb{A}^1_\Bbbk$ although the arguments below are valid over more general one-dimensional bases.

Suppose $\tilde{Y} \to \mathbb{A}^1_\Bbbk$ is a flat deformation of a variety Y and denote by $j : Y \to \tilde{Y}$ the inclusion of Y as the fibre over $0 \in \mathbb{A}^1_\Bbbk$. We assume, as above, that we have a compatible $\mathbb{C}^\times$-action on $\tilde{Y}$ (i.e. it maps fibres to fibres and acts on the base $\mathbb{A}^1_\Bbbk$ by $x \mapsto t^2 x$).

Given an object $\mathcal{G} \in D(Y)$ we get the standard exact triangle

$$\mathcal{G}[1] \otimes N^\vee_{Y/\tilde{Y}} \to j^* j_* \mathcal{G} \to \mathcal{G}$$

obtained via the natural adjunction maps. Now $N_{Y/\tilde{Y}} \cong \mathcal{O}_Y\{2\}$ the connecting morphism gives an endomorphism $\alpha : \mathcal{G}[-1]\{1\} \to \mathcal{G}[1]\{-1\}$. Alternatively, α can be defined as the composition $\mathrm{At}(\mathcal{G}) \cdot \kappa(j) \in \mathrm{Ext}^2(\mathcal{G},\mathcal{G})$ where $\mathrm{At}(\mathcal{G}) \in \mathrm{Ext}^1(\mathcal{G}, \mathcal{G} \otimes \Omega_Y)$ is the Atiyah class of $\mathcal{G}$ and $\kappa(j) \in \mathrm{Ext}^1(\Omega_Y, N_{Y/\tilde{Y}})$ is the Kodaira-Spencer class (see the Appendix of [HT] for a proof of the equivalence of these two definitions). From either definition it is apparent that $j : Y \to \tilde{Y}$ only defines the map α up to a non-zero multiple because, though $N_{Y/\tilde{Y}} \cong \mathcal{O}_Y\{2\}$, this isomorphism is not canonical. Nevertheless, regardless of the value of this non-zero multiple we will always have

$$\mathrm{Cone}(\mathcal{G}[-1]\{1\} \xrightarrow{\alpha} \mathcal{G}[1]\{-1\}) \cong j^* j_* \mathcal{G}\{1\}.$$

If we have flat deformations $\tilde{Y}_1 \to \mathbb{A}^1_\Bbbk$ and $\tilde{Y}_2 \to \mathbb{A}^1_\Bbbk$ of varieties Y_1 and Y_2 with compatible $\mathbb{C}^\times$-actions then we get deformations $\tilde{Y}_1 \times Y_2$ and $Y_1 \times \tilde{Y}_2$ of $Y_1 \times Y_2$ and subsequently maps $\alpha_1, \alpha_2 : \mathcal{G}_{12}[-1]\{1\} \to \mathcal{G}_{12}[1]\{-1\}$ for any $\mathcal{G}_{12} \in D(Y_1 \times Y_2)$. As before α_1 and α_2 are only defined up to a non-zero multiple. Thus we get a family of maps

$$\mathcal{G}_{12}[-1]\{1\} \xrightarrow{a\alpha_1 + b\alpha_2} \mathcal{G}_{12}[1]\{-1\}$$

where $a, b \in \Bbbk$. The cone of $a\alpha_1 + b\alpha_2$ will be isomorphic to $j_{12}^* j_{12*} \mathcal{G}_{12}\{1\}$ where $j_{12} : Y_1 \times Y_2 \to \tilde{Y}_1 \tilde{\times}_\Bbbk \tilde{Y}_2$ is the inclusion into some (twisted) fibre product of $\tilde{Y}_1$ and $\tilde{Y}_2$ (the parameters a, b control the twisting).

Suppose we are in the special case $Y_1 = Y_2$. Let Y_3 be a third variety. Given $\mathcal{G}_{23} \in D(Y_2 \times Y_3)$ we would like to compute

$$\mathcal{G}_{23} * \mathrm{Cone}(\mathcal{O}_\Delta[-1]\{1\} \xrightarrow{a\alpha_1 + b\alpha_2} \mathcal{O}_\Delta[1]\{-1\}) \cong \mathcal{G}_{23} * (j_{12}^* j_{12*} \mathcal{O}_\Delta\{1\}).$$

To do this we will use the following result.

Lemma 4.1. *Given three spaces Y_i $(i = 1, 2, 3)$ with deformations $\tilde{Y}_i \to \mathbb{A}^1_{\Bbbk}$ $(i = 1, 2, 3)$ denote by $j_{12} : Y_1 \times Y_2 \to \tilde{Y}_1 \times_{\mathbb{A}^1_{\Bbbk}} \tilde{Y}_2$, $j_{23} : Y_2 \times Y_3 \to \tilde{Y}_2 \times_{\mathbb{A}^1_{\Bbbk}} \tilde{Y}_3$ and $j_{13} : Y_1 \times Y_3 \to \tilde{Y}_1 \times_{\mathbb{A}^1_{\Bbbk}} \tilde{Y}_3$ the natural inclusions. Given $\mathcal{G}_{12} \in D(Y_1 \times Y_2)$ and $\mathcal{G}_{23} \in D(Y_2 \times Y_3)$ we have*

$$j_{13*} \left(\mathcal{G}_{23} * (j_{12}^* j_{12*} \mathcal{G}_{12})\right) \cong (j_{23*}\mathcal{G}_{23}) * (j_{12*}\mathcal{G}_{12}) \cong j_{13*} \left((j_{23}^* j_{23*}\mathcal{G}_{23}) * \mathcal{G}_{12}\right)$$

where the convolution product $*$ in the middle term is relative to the base $\mathbb{A}^1_{\Bbbk}$. In simplified notation this says

$$j_*(\mathcal{G} * (j^* j_* \mathcal{G})) \cong (j_*\mathcal{G}) * (j_*\mathcal{G}) \cong j_*((j^* j_*\mathcal{G}) * \mathcal{G})$$

which is reminiscent of the projection formula. Everything still holds if we also have a compatible $\mathbb{C}^\times$-action.

Proof. The proof amounts to diagram chasing.

$$\begin{aligned} j_{13*} \left(\mathcal{G}_{23} * j_{12}^* j_{12*}\mathcal{G}_{12}\right) &\cong j_{13*}\pi_{13*} \left(\pi_{12}^*(j_{12}^* j_{12*}\mathcal{G}_{12}) \otimes \pi_{23}^*\mathcal{G}_{23}\right) \\ &\cong \tilde{\pi}_{13*}\tilde{j}_* \left(\tilde{j}^*\tilde{\pi}_{12}^* j_{12*}\mathcal{G}_{12} \otimes \pi_{23}^*\mathcal{G}_{23}\right) \end{aligned}$$

where we use the commuting squares

$$\begin{array}{ccc} Y_1 \times Y_2 \times Y_3 & \xrightarrow{\tilde{j}} & \tilde{Y}_1 \times_{\mathbb{A}^1_{\Bbbk}} \tilde{Y}_2 \times_{\mathbb{A}^1_{\Bbbk}} \tilde{Y}_3 \\ \downarrow{\scriptstyle \pi_{12}} & & \downarrow{\scriptstyle \tilde{\pi}_{12}} \\ Y_1 \times Y_2 & \xrightarrow{j_{12}} & \tilde{Y}_1 \times_{\mathbb{A}^1_{\Bbbk}} \tilde{Y}_2 \end{array} \qquad \begin{array}{ccc} Y_1 \times Y_2 \times Y_3 & \xrightarrow{\tilde{j}} & \tilde{Y}_1 \times_{\mathbb{A}^1_{\Bbbk}} \tilde{Y}_2 \times_{\mathbb{A}^1_{k}} \tilde{Y}_3 \\ \downarrow{\scriptstyle \pi_{13}} & & \downarrow{\scriptstyle \tilde{\pi}_{13}} \\ Y_1 \times Y_3 & \xrightarrow{j_{13}} & \tilde{Y}_1 \times_{\mathbb{A}^1_{\Bbbk}} \tilde{Y}_3 \end{array}$$

The projection formula gives

$$\begin{aligned} \tilde{\pi}_{13*}\tilde{j}_* \left(\tilde{j}^*\tilde{\pi}_{12}^* j_{12*}\mathcal{G}_{12} \otimes \pi_{23}^*\mathcal{G}_{23}\right) &\cong \tilde{\pi}_{13*} \left(\tilde{\pi}_{12}^*(j_{12*}\mathcal{G}_{12}) \otimes \tilde{j}_*\pi_{23}^*\mathcal{G}_{23}\right) \\ &\cong \tilde{\pi}_{13*} \left(\tilde{\pi}_{12}^*(j_{12*}\mathcal{G}_{12}) \otimes \tilde{\pi}_{23}^*(j_{23*}\mathcal{G}_{23})\right) \\ &\cong j_{23*}\mathcal{G}_{23} * j_{12*}\mathcal{G}_{12} \end{aligned}$$

where the second line follows from flat base change on the commuting square

$$\begin{array}{ccc} Y_1 \times Y_2 \times Y_3 & \xrightarrow{\tilde{j}} & \tilde{Y}_1 \times_{\mathbb{A}^1_{\Bbbk}} \tilde{Y}_2 \times_{\mathbb{A}^1_{\Bbbk}} \tilde{Y}_3 \\ \downarrow{\scriptstyle \pi_{23}} & & \downarrow{\scriptstyle \tilde{\pi}_{23}} \\ Y_2 \times Y_3 & \xrightarrow{j_{23}} & \tilde{Y}_2 \times_{\mathbb{A}^1_{\Bbbk}} \tilde{Y}_3 \end{array}$$

This proves the first isomorphism in the Lemma. The second isomorphism follows similarly. If we also have a compatible $\mathbb{C}^\times$-action nothing changes in the proof since all the maps are naturally $\mathbb{C}^\times$-equivariant. □

Corollary 4.2. *We follow the notation above with $Y_1 = Y_2$, $\tilde{Y}_1 = \tilde{Y}_2$ and $\mathcal{G}_{23} \in D(Y_2 \times Y_3)$. If*

$$\mathrm{Ext}^{-1}(\mathcal{G}_{23}, \mathcal{G}_{23}\{2\}) = 0 \text{ and } \mathrm{End}(\mathcal{G}_{23}) \cong \Bbbk \cdot \mathrm{I}$$

then

$$\mathcal{G}_{23} * \mathrm{Cone}(\mathcal{O}_\Delta[-1]\{1\} \xrightarrow{a\alpha_1 + b\alpha_2} \mathcal{O}_\Delta[1]\{-1\}) \cong j_{23}^* j_{23*}\mathcal{G}_{23}\{1\}.$$

for sufficiently general choices of $a, b \in \Bbbk$. Here j_{23} is the inclusion $Y_2 \times Y_3 \to \tilde{Y}_2 \times Y_3$ and by sufficiently general we mean the complement of a hyperplane.

Proof. Fix a choice $\phi : \mathcal{G}_{23}[-1]\{1\} \to \mathcal{G}_{23}[1]\{-1\}$ of the connecting morphism for j_{23}. So the cone of any non-zero multiple of ϕ is isomorphic to $j_{23}^* j_{23*} \mathcal{G}_{23}\{1\}$.

Denote $Y_1 = Y_2$ by Y and Y_3 by Y'. First consider the case $a = 0$ and let

$$\beta_2 = \mathcal{G}_{23} * \alpha_2 : \mathcal{G}_{23}[-1]\{1\} \to \mathcal{G}_{23}[1]\{-1\}.$$

Then

$$\begin{aligned} j_{13*}\left(\mathcal{G}_{23} * \mathrm{Cone}(\mathcal{O}_\Delta[-1]\{1\} \xrightarrow{\alpha_2} \mathcal{O}_\Delta[1]\{-1\})\right) &\cong j_{13*}\left(\mathcal{G}_{23} * j_{12}^* j_{12*} \mathcal{O}_\Delta\{1\}\right) \\ &\cong j_{13*}\left(j_{23}^* j_{23*} \mathcal{G}_{23} * \mathcal{O}_\Delta\{1\}\right) \\ &\cong j_{13*} j_{23}^* j_{23*} \mathcal{G}_{23}\{1\}. \end{aligned}$$

where, since $a = 0$, $j_{12} : Y \times Y \to Y \times \tilde{Y}$. Then $j_{13} : Y \times Y' \to Y \times Y' \times \mathbb{A}^1_{\Bbbk}$ is the inclusion into a trivial family so $j_{13}^* j_{13*}(\cdot) \cong (\cdot) \oplus (\cdot)[1]\{-2\}$. Pulling back by j_{13}^* we get

$$\left(\mathcal{G}_{23} * \mathrm{Cone}(\mathcal{O}_\Delta[-1]\{1\} \xrightarrow{\alpha_2} \mathcal{O}_\Delta[1]\{-1\})\right) \otimes_{\Bbbk} (\Bbbk \oplus \Bbbk[1]\{-2\}) \cong j_{23}^* j_{23*} \mathcal{G}_{23}\{1\} \otimes_{\Bbbk} (\Bbbk \oplus \Bbbk[1]\{-2\}).$$

Since $D(Y \times Y')$ is idempotent complete this implies

$$\mathcal{G}_{23} * \mathrm{Cone}(\mathcal{O}_\Delta[-1]\{1\} \xrightarrow{\alpha_2} \mathcal{O}_\Delta[1]\{-1\}) \cong j_{23}^* j_{23*} \mathcal{G}_{23}\{1\}$$

(see Remark 2.1). Thus

$$\mathrm{Cone}(\mathcal{G}_{23}[-1]\{1\} \xrightarrow{\beta_2} \mathcal{G}_{23}[1]\{-1\}) \cong j_{23}^* j_{23*} \mathcal{G}_{23}\{1\}.$$

Then by Lemma 4.3 below this means $\beta_2 = u\phi$ for some non-zero u.

Now consider the case $b = 0$ and let $\beta_1 = \mathcal{G}_{23} * \alpha_1$.

The same argument as above shows that

$$j_{13*}\left(\mathcal{G}_{23} * \mathrm{Cone}(\mathcal{O}_\Delta[-1]\{1\} \xrightarrow{\alpha_1} \mathcal{O}_\Delta[1]\{-1\})\right) \cong j_{13*} j_{23}^* j_{23*} \mathcal{G}_{23}\{1\}$$

where $j_{12} : Y \times Y \to \tilde{Y} \times Y'$. Now, since $b = 0$, $j_{23} : Y \times Y' \to Y \times Y' \times \mathbb{A}^1_{\Bbbk}$ is the inclusion into the trivial family. Thus $j_{23}^* j_{23*} \mathcal{G}_{23} \cong \mathcal{G}_{23} \oplus \mathcal{G}_{23}[1]\{-2\}$ so that

$$j_{13*}\left(\mathcal{G}_{23} * \mathrm{Cone}(\mathcal{O}_\Delta[-1]\{1\} \xrightarrow{\alpha_1} \mathcal{O}_\Delta[1]\{-1\})\right) \cong j_{13*}\mathcal{G}_{23} \oplus j_{13*}\mathcal{G}_{23}[1].$$

This means $j_{13*}(\beta_1) = 0$ so that by Lemma 4.3 below we get that β_1 is some multiple of the connecting morphism for $j_{13} : Y \times Y' \to \tilde{Y} \times Y'$.

Thus $\beta_1 = v\phi$ and hence $\mathcal{G}_{23} * (a\alpha_1 + b\alpha_2) = (au + bv)\phi$ where $u \neq 0$. So for general a, b,

$$\mathcal{G}_{23} * \mathrm{Cone}(\mathcal{O}_\Delta[-1]\{1\} \xrightarrow{a\alpha_1 + b\alpha_2} \mathcal{O}_\Delta[1]\{-1\}) \cong j_{23}^* j_{23*} \mathcal{G}_{23}\{1\}$$

as desired.

Lemma 4.3. *Let Y be a variety and suppose $\mathcal{G} \in D(Y)$ with*

$$\mathrm{Ext}^{-1}(\mathcal{G}, \mathcal{G}\{2\}) = 0 \text{ and } \mathrm{End}(\mathcal{G}) \cong \Bbbk \cdot \mathrm{I}.$$

Let $\tilde{Y} \to \mathbb{A}^1_{\Bbbk}$ be a flat deformation of Y carrying a compatible $\mathbb{C}^\times$-action. Denote by $j : Y \to \tilde{Y}$ the natural inclusion and let $\beta \in \mathrm{Hom}(\mathcal{G}[-1]\{1\}, \mathcal{G}[1]\{-1\})$ be some map. Then

- *if $\mathrm{Cone}(\beta) \cong j^* j_* \mathcal{G}\{1\}$ then β is some non-zero multiple of the connecting morphism corresponding to j*
- *if $j_*\beta = 0$ then β is some multiple of the connecting morphism corresponding to j*

Proof. Consider the following diagram

$$\begin{array}{ccccccc} \mathcal{G}[-1]\{1\} & \xrightarrow{\beta} & \mathcal{G}[1]\{-1\} & \longrightarrow & \mathrm{Cone}(\beta) & \longrightarrow & \mathcal{G}\{1\} \\ \downarrow & & \downarrow & & \downarrow \sim & & \downarrow \\ \mathcal{G}[-1]\{1\} & \xrightarrow{\gamma} & \mathcal{G}[1]\{-1\} & \longrightarrow & j^*j_*\mathcal{G}\{1\} & \longrightarrow & \mathcal{G}\{1\} \end{array}$$

where the dashed lines denote maps which are to be determined. Since $\mathrm{Ext}^{-1}(\mathcal{G}, \mathcal{G}\{2\}) = 0$ the composition

$$\mathcal{G}[1]\{-1\} \to \mathrm{Cone}(\beta) \xrightarrow{\sim} j^*j_*\mathcal{G}\{1\} \to \mathcal{G}\{1\}$$

is zero. This means that we can fill all the dashed arrows with maps making the diagram commute (this holds very generally in any triangulated category and follows by studying associated long exact sequences). These maps we fill in must be non-zero and since $\mathrm{End}(\mathcal{G}) = \Bbbk \cdot \mathrm{I}$ they must be isomorphisms. Thus β is, up to a non-zero multiple, equal to the connecting morphism γ. This proves the first claim.

For the second claim notice that the map $j_*\mathcal{G}[-1]\{1\} \xrightarrow{j_*\beta} j_*\mathcal{G}[1]\{-1\}$ is adjoint to the composition

$$j^*j_*\mathcal{G}[-1]\{1\} \xrightarrow{\varepsilon} \mathcal{G}[-1]\{1\} \xrightarrow{\beta} \mathcal{G}[1]\{-1\}$$

where ε is the counit of adjunction. So if $j_*\beta = 0$ then this composition is zero. But now if we consider the diagram

$$\begin{array}{ccccc} j^*j_*\mathcal{G}[-1]\{1\} & \xrightarrow{\varepsilon} & \mathcal{G}[-1]\{1\} & \xrightarrow{\gamma} & \mathcal{G}[1]\{-1\} \\ \downarrow & & \mathrm{I} \downarrow & & \downarrow \\ \mathrm{Cone}(\beta)[-1] & \longrightarrow & \mathcal{G}[-1]\{1\} & \xrightarrow{\beta} & \mathcal{G}[1]\{-1\} \end{array}$$

this means that we can again find morphisms to fill in the dashed arrows and make the diagram commute. Since $\mathrm{End}(\mathcal{G}) = \Bbbk \cdot \mathrm{I}$, β is some multiple of γ as desired. □

□

If we have a geometric categorical $\mathbb{C}^\times$-equivariant $\mathfrak{sl}_2$ action then $\tilde{Y}(\lambda)$ induces two deformations on $Y(\lambda) \times Y(\lambda)$ (one on each factor) and subsequently two maps $\alpha_1, \alpha_2 : \mathcal{O}_\Delta[-1]\{1\} \to \mathcal{O}_\Delta[1]\{-1\}$. Now consider $\mathcal{E}(\lambda+1) \in D(Y(\lambda) \times Y(\lambda+2))$. By Corollary 4.2 for sufficiently general $a, b \in \Bbbk$ the map

$$\theta(\lambda) := a\alpha_1 + b\alpha_2 : \mathcal{O}_\Delta[-1]\{1\} \to \mathcal{O}_\Delta[1]\{-1\}$$

satisfies

$$\mathcal{E}(\lambda+1) * \mathrm{Cone}(\mathcal{O}_\Delta[-1]\{1\} \xrightarrow{\theta(\lambda)} \mathcal{O}_\Delta[1]\{-1\}) \cong j^*j_*\mathcal{E}(\lambda+1)\{1\}$$

where j is the inclusion $Y(\lambda) \times Y(\lambda+2) \to \tilde{Y}(\lambda) \times Y(\lambda+2)$.

Note that the same argument shows that for sufficiently general $a, b \in \Bbbk$ the map $\theta(\lambda)$ also satisfies

$$\mathrm{Cone}(\mathcal{O}_\Delta[-1]\{1\} \xrightarrow{\theta(\lambda)} \mathcal{O}_\Delta[1]\{-1\}) * \mathcal{E}(\lambda-1) \cong j'^*j'_*\mathcal{E}(\lambda-1)\{1\}$$

where j' is the inclusion $Y(\lambda-2) \times Y(\lambda) \to Y(\lambda-2) \times \tilde{Y}(\lambda)$.

Definition 4.4. *For each λ we fix $\theta(\lambda)$ such that:*

$$\mathcal{E}(\lambda+1) * \mathrm{Cone}(\mathcal{O}_\Delta[-1]\{1\} \xrightarrow{\theta(\lambda)} \mathcal{O}_\Delta[1]\{-1\}) \cong j^*j_*\mathcal{E}(\lambda+1)\{1\}$$

where $j : Y(\lambda) \times Y(\lambda+2) \to \tilde{Y}(\lambda) \times Y(\lambda+2)$ and

$$\mathrm{Cone}(\mathcal{O}_\Delta[-1]\{1\} \xrightarrow{\theta(\lambda)} \mathcal{O}_\Delta[1]\{-1\}) * \mathcal{E}(\lambda-1) \cong j'^*j'_*\mathcal{E}(\lambda-1)\{1\}$$

where $j' : Y(\lambda-2) \times Y(\lambda) \to Y(\lambda-2) \times \tilde{Y}(\lambda)$.

4.2. **Formality of $\mathsf{E}^{(r_2)} \circ \mathsf{E}^{(r_1)} \cong \mathsf{E}^{(r_1+r_2)} \otimes H^\star(\mathbb{G}(r_1, r_1+r_2))$.** Denote by

$$\Theta(\lambda-1+r) : \mathcal{E}(\lambda+r) * \mathcal{E}^{(r)}(\lambda-1)[-1]\{1\} \rightarrow \mathcal{E}(\lambda+r) * \mathcal{E}^{(r)}(\lambda-1)[1]\{-1\}$$

the map induced by $\theta(\lambda-1+r)$ via

$$\mathcal{E}(\lambda+r) * (\mathcal{O}_\Delta[-1]\{1\} \xrightarrow{\theta(\lambda-1+r)} \mathcal{O}_\Delta[1]\{-1\}) * \mathcal{E}^{(r)}(\lambda-1).$$

This map has the following property.

Lemma 4.5. *For $i = 0, \dots, r$ the map*

$$\Theta(\lambda-1+r)^i : \mathcal{E}(\lambda+r) * \mathcal{E}^{(r)}(\lambda-1)[-1]\{1\} \rightarrow \mathcal{E}(\lambda+r) * \mathcal{E}^{(r)}(\lambda-1)[-1+2i]\{1-2i\}$$

induces an isomorphism

$$\mathcal{H}^{-r}(\mathcal{E}(\lambda+r) * \mathcal{E}^{(r)}(\lambda-1)) \cong \mathcal{E}^{(r+1)}(\lambda)\{-r\} \rightarrow \mathcal{H}^{-r+2i}(\mathcal{E}(\lambda+r) * \mathcal{E}^{(r)}(\lambda-1))\{-2i\}.$$

Proof. Since we have a geometric categorical $\mathbb{C}^\times$-equivariant $\mathfrak{sl}_2$ action we know that

$$\mathcal{H}^*(\mathcal{E}(\lambda+r) * \mathcal{E}^{(r)}(\lambda-1)) \cong \mathcal{E}^{(r+1)}(\lambda) \otimes_{\Bbbk} H^\star(\mathbb{P}^r).$$

Now consider the exact triangle

$$\mathcal{E}(\lambda+r) * \mathcal{E}^{(r)}(\lambda-1)[-1]\{1\} \xrightarrow{\Theta(\lambda-1+r)} \mathcal{E}(\lambda+r) * \mathcal{E}^{(r)}(\lambda-1)[1]\{-1\} \rightarrow \mathrm{Cone}(\Theta(\lambda-1+r))$$

and the induced long exact sequence

$$\dots \mathcal{H}^{i-1}(\mathcal{E}(\lambda+r) * \mathcal{E}^{(r)}(\lambda-1))\{1\} \xrightarrow{\gamma_i} \mathcal{H}^{i+1}(\mathcal{E}(\lambda+r) * \mathcal{E}^{(r)}(\lambda-1))\{-1\} \rightarrow \mathcal{H}^i(\mathrm{Cone}\Theta(\lambda-1+r)) \rightarrow \dots.$$

If we could show that $\mathrm{Cone}(\Theta(\lambda-1+r))$ has cohomology only in degrees $-r-1$ and r then the maps γ_i would have to be isomorphisms for $i = -r+1, -r+3, \dots, r-1$. So applying $\Theta(\lambda-1+r)^i$ we would get an induced map

$$\mathcal{H}^{-r}(\mathcal{E}(\lambda+r) * \mathcal{E}^{(r)}(\lambda-1)) \xrightarrow{\gamma} \mathcal{H}^{-r+2i}(\mathcal{E}(\lambda+r) * \mathcal{E}^{(r)}(\lambda-1))\{-2i\}$$

where $\gamma = \gamma_{-r+2i-1} \circ \dots \circ \gamma_{-r+1}$, which is an isomorphism.

Now $\mathrm{Cone}(\Theta(\lambda-1+r))$ is by definition

$$\mathcal{E}(\lambda+r) * \mathrm{Cone}(\mathcal{O}_\Delta[-1]\{1\} \xrightarrow{\theta(\lambda-1+r)} \mathcal{O}_\Delta[1]\{-1\}) * \mathcal{E}^{(r)}(\lambda-1)$$

which, by the way we defined $\theta(\lambda-1+r)$, equals

$$j_{23}^* j_{23*} \mathcal{E}(\lambda+r) * \mathcal{E}^{(r)}(\lambda-1)\{1\}$$

where $j_{23} : Y(\lambda-1+r) \times Y(\lambda+1+r) \rightarrow \tilde{Y}(\lambda-1+r) \times Y(\lambda+1+r)$.

Note that it suffices to show that $j_{13*}(j_{23}^* j_{23*} \mathcal{E}(\lambda+r) * \mathcal{E}^{(r)}(\lambda-1))$ has cohomology only in degrees $-r-1$ and r where $j_{13} : Y(\lambda-1-r) \times Y(\lambda+1+r) \rightarrow Y(\lambda-1-r) \times Y(\lambda+1+r) \times \mathbb{A}^1_{\Bbbk}$. By Lemma 4.1

$$\begin{aligned}
\mathcal{H}^*(j_{13*}(j_{23}^* j_{23*} \mathcal{E}(\lambda+r) * \mathcal{E}^{(r)}(\lambda-1)) &\cong \mathcal{H}^*(j_{23*}\mathcal{E}(\lambda+r) * j_{12*}\mathcal{E}^{(r)}(\lambda-1)) \\
&\cong j_{13*}\mathcal{E}^{(r+1)}(\lambda) \otimes_{\Bbbk} (\Bbbk[-r]\{r\} \oplus \Bbbk[r+1]\{-r-2\})
\end{aligned}$$

and we are done. ∎

Now define

$$\iota : \mathcal{E}^{(r+1)}(\lambda)[r]\{-r\} \rightarrow \mathcal{E}(\lambda+r) * \mathcal{E}^{(r)}(\lambda-1)$$

as the natural inclusion into the bottom. Note that to define this inclusion we only need to know that

$$\mathcal{H}^\star(\mathcal{E}(\lambda+r) * \mathcal{E}^{(r)}(\lambda-1)) \cong \mathcal{E}^{(r+1)}(\lambda) \otimes_{\Bbbk} H^\star(\mathbb{P}^r)$$

at the level of cohomology. The fact that $\text{Ext}^i(\mathcal{E}^{(r+1)}(\lambda), \mathcal{E}^{(r+1)}(\lambda)\{j\}) = 0$ for $i < 0$ (and any $j \in \mathbb{Z}$) while $\text{End}(\mathcal{E}^{(r+1)}(\lambda)) \cong \Bbbk \cdot \text{I}$ means that ι is actually unique (up to a non-zero multiple). Similarly we can define

$$\pi : \mathcal{E}(\lambda + r) * \mathcal{E}^{(r)}(\lambda - 1) \to \mathcal{E}^{(r+1)}(\lambda)[-r]\{r\}$$

as the natural projection out of the top.

Corollary 4.6. *We have*

$$\mathcal{E}^{(r_2)}(\lambda + r_1) * \mathcal{E}^{(r_1)}(\lambda - r_2) \cong \mathcal{E}^{(r_1+r_2)}(\lambda) \otimes_\Bbbk H^\star(\mathbb{G}(r_1, r_1 + r_2)).$$

This relation also holds if we replace $\mathcal{E}$ by $\mathcal{F}$.

Proof. We know this isomorphism at the level of cohomology and now we want to show formality (i.e. that the complex breaks up as a direct sum of its cohomology).

We first deal with the case $r_1 = r$ and $r_2 = 1$ as follows. Consider the map

$$(\Theta(\lambda - 1 + r)^r \circ \iota[-2r]\{2r\}) \oplus \cdots \oplus (\Theta(\lambda - 1 + r) \circ \iota[-2]\{2\}) \oplus \iota :$$
$$\bigoplus_{i=0}^{r} \mathcal{E}^{(r+1)}(\lambda)[r - 2i]\{-r + 2i\} \to \mathcal{E}(\lambda + r) * \mathcal{E}^{(r)}(\lambda - 1).$$

By Lemma 4.5 this is an isomorphism on cohomology and we are done (a quasi-isomorphism is an isomorphism in the derived category).

Applying this isomorphism repeatedly we find that

$$\mathcal{E}(\lambda + r_1 + r_2 - 1) * \cdots * \mathcal{E}(\lambda - r_1 - r_2 + 1) \cong \mathcal{E}^{(r_1+r_2)}(\lambda) \otimes_\Bbbk H^\star(\mathbb{F}l_{r_1+r_2})$$

where $\mathbb{F}l_r$ denotes the complete flag of $\mathbb{C}^r$. Similarly, one finds that

$$\mathcal{E}(\lambda + r_1 - r_2 - 1) * \cdots * \mathcal{E}(\lambda - r_1 - r_2 + 1) \cong \mathcal{E}^{(r_1)}(\lambda - r_2) \otimes_\Bbbk H^\star(\mathbb{F}l_{r_1})$$

and

$$\mathcal{E}(\lambda + r_1 + r_2 - 1) * \cdots * \mathcal{E}(\lambda + r_1 - r_2 + 1) \cong \mathcal{E}^{(r_2)}(\lambda + r_1) \otimes_\Bbbk H^\star(\mathbb{F}l_{r_2}).$$

Thus, using that $D(Y(\lambda - r_1 - r_2) \times Y(\lambda + r_1 + r_2))$ is idempotent complete (see Remark 2.1), $\mathcal{E}^{(r_2)}(\lambda + r_1) * \mathcal{E}^{(r_1)}(\lambda - r_2)$ breaks up as a direct sum of (shifted) copies of $\mathcal{E}^{(r_1+r_2)}(\lambda)$ (here we are using that all the $\mathcal{E}$'s are simple).

To figure out how many copies and in which degrees one notes that

$$H^\star(\mathbb{F}l_{r_1+r_2}) \cong H^\star(\mathbb{F}l_{r_1}) \otimes_\Bbbk H^\star(\mathbb{G}(r_1, r_2 + r_2)) \otimes_\Bbbk H^\star(\mathbb{F}l_{r_2}).$$

The analogous result for $\mathcal{F}$s follows by taking adjoints. □

4.3. **Formality of $\mathsf{F} \circ \mathsf{E} \cong \mathsf{E} \circ \mathsf{F} \oplus \mathrm{I} \otimes H^\star(\mathbb{P})$.** The proof of formality here is analogous to the one in the last section. We suppose $\lambda \leq 0$ and begin with a simple observation.

Lemma 4.7. *If $\lambda \leq 0$ we have*

$$\text{Hom}(\mathcal{P}, \mathcal{E}(\lambda - 1) * \mathcal{F}(\lambda - 1)) = 0 \text{ and } \text{Hom}(\mathcal{E}(\lambda - 1) * \mathcal{F}(\lambda - 1), \mathcal{P}) = 0$$

where $\mathcal{H}^*(\mathcal{P}) \cong \mathcal{O}_\Delta \otimes_\Bbbk H^\star(\mathbb{P}^{-\lambda-1})$.

Proof. We have

$$\begin{aligned} \text{Hom}(\mathcal{O}_\Delta[n]\{-n\}, \mathcal{E}(\lambda - 1) * \mathcal{F}(\lambda - 1)) &\cong \text{Hom}(\mathcal{E}(\lambda - 1)_L[n]\{-n\}, \mathcal{F}(\lambda - 1)) \\ &\cong \text{Hom}(\mathcal{F}(\lambda - 1)[-\lambda + 1 + n]\{\lambda - 1 - n\}, \mathcal{F}(\lambda - 1)) \end{aligned}$$

which is zero if $n > \lambda - 1$. Since

$$\mathcal{H}^*(\mathcal{P}) \cong \bigoplus_{j=0}^{-\lambda-1} \mathcal{O}_\Delta[-\lambda - 1 - 2j]\{\lambda + 1 + 2j\}$$

this means $\mathrm{Hom}(\mathcal{P}, \mathcal{E}(\lambda-1) * \mathcal{F}(\lambda-1)) = 0$.

Similarly, we have

$$\begin{aligned} \mathrm{Hom}(\mathcal{E}(\lambda-1) * \mathcal{F}(\lambda-1), \mathcal{O}_\Delta[n]\{-n\}) &\cong \mathrm{Hom}(\mathcal{F}(\lambda-1), \mathcal{E}(\lambda-1)_R[n]\{-n\}) \\ &\cong \mathrm{Hom}(\mathcal{F}(\lambda-1), \mathcal{F}(\lambda-1)[\lambda-1+n]\{-\lambda+1-n\}) \end{aligned}$$

which is zero if $n < -\lambda+1$. This means $\mathrm{Hom}(\mathcal{E}(\lambda-1) * \mathcal{F}(\lambda-1), \mathcal{P}) = 0$. □

Recall that

$$\mathcal{F}(\lambda+1) * \mathcal{E}(\lambda+1) \cong \mathcal{E}(\lambda-1) * \mathcal{F}(\lambda-1) \oplus \mathcal{P}.$$

Lemma 4.7 means that given a map $\mathcal{F}(\lambda+1)*\mathcal{E}(\lambda+1) \to \mathcal{F}(\lambda+1)*\mathcal{E}(\lambda+1)[i]\{j\}$ there is an induced map $\mathcal{P} \to \mathcal{P}[i]\{j\}$ well defined up to a non-zero multiple.

We denote by

$$\Theta(\lambda+2) : \mathcal{F}(\lambda+1) * \mathcal{E}(\lambda+1)[-1]\{1\} \to \mathcal{F}(\lambda+1) * \mathcal{E}(\lambda+1)[1]\{-1\}$$

the map induced by $\theta(\lambda+2)$ via

$$\mathcal{F}(\lambda+1) * (\mathcal{O}_\Delta[-1]\{1\} \xrightarrow{\theta(\lambda+2)} \mathcal{O}_\Delta[1]\{-1\}) * \mathcal{E}(\lambda+1).$$

Lemma 4.8. *For $i = 0, \ldots, -\lambda-1$ the map*

$$\Theta(\lambda+2)^i : \mathcal{F}(\lambda+1) * \mathcal{E}(\lambda+1)[-1]\{1\} \to \mathcal{F}(\lambda+1) * \mathcal{E}(\lambda+1)[-1+2i]\{1-2i\}$$

induces a morphism $\mathcal{P}[-1]\{1\} \to \mathcal{P}[-1+2i]\{1-2i\}$ which induces an isomorphism

$$\mathcal{H}^{\lambda+1}(\mathcal{P}) \cong \mathcal{O}_\Delta\{\lambda+1\} \to \mathcal{H}^{\lambda+1+2i}(\mathcal{P})\{-2i\} \cong \mathcal{O}_\Delta\{\lambda+1\}.$$

Proof. Since we have a geometric categorical $\mathbb{C}^\times$-equivariant $\mathfrak{sl}_2$ action we know that

$$\mathcal{F}(\lambda+1) * \mathcal{E}(\lambda+1) \cong \mathcal{E}(\lambda-1) * \mathcal{F}(\lambda-1) \oplus \mathcal{P}$$

where $\mathcal{H}^*(\mathcal{P}) \cong \mathrm{I} \otimes_{\Bbbk} H^\star(\mathbb{P}^{-\lambda-1})$.

Now consider the exact triangle

$$\mathcal{F}(\lambda+1) * \mathcal{E}(\lambda+1)[-1]\{1\} \xrightarrow{\Theta(\lambda+2)} \mathcal{F}(\lambda+1) * \mathcal{E}(\lambda+1)[1]\{-1\} \to \mathrm{Cone}(\Theta(\lambda+2))$$

and the induced long exact sequence

(4)
$$\cdots \to \mathcal{H}^{i-1}(\mathcal{E}(\lambda-1)*\mathcal{F}(\lambda-1)\oplus\mathcal{P})\{1\} \xrightarrow{\gamma_i} \mathcal{H}^{i+1}(\mathcal{E}(\lambda-1)*\mathcal{F}(\lambda-1)\oplus\mathcal{P})\{-1\} \to \mathcal{H}^i(\mathrm{Cone}\Theta(\lambda+2)) \to \ldots$$

On the other hand we also know that

$$j_*\mathrm{Cone}(\Theta(\lambda+2)) \cong j_*(\mathcal{F}(\lambda+1) * j_{12}^* j_{12*}\mathcal{E}(\lambda+1)\{1\}) \cong j_{12*}\mathcal{F}(\lambda+1) * j_{12*}\mathcal{E}(\lambda+1)\{1\}$$

where $j_{12} : Y(\lambda) \times Y(\lambda+2) \to Y(\lambda) \times \tilde{Y}(\lambda+2)$, j is the inclusion into the trivial deformation and the $*$ convolution on the right is relative to the base $\mathbb{A}^1_{\Bbbk}$. Since j is the trivial inclusion this means

$$\mathrm{Cone}(\Theta(\lambda+2)) \cong j_{12*}\mathcal{F}(\lambda+1) * j_{12*}\mathcal{E}(\lambda+1)\{1\}$$

where the $*$ convolution on the right is no longer relative to the base. Now, because we have a geometric categorical $\mathfrak{sl}_2$ action we also get the exact triangle

(5)
$$\mathcal{E}(\lambda-1)*\mathcal{F}(\lambda-1)[1]\{-2\}\oplus\mathcal{O}_\Delta[-\lambda]\{\lambda-1\} \to \mathrm{Cone}(\Theta(\lambda+2)) \to \mathcal{E}(\lambda-1)*\mathcal{F}(\lambda-1)\oplus\mathcal{O}_\Delta[\lambda+1]\{-\lambda-1\}.$$

Now let's compare the long exact sequence induced by the exact triangle (5) to the long exact sequence from (4). If any map γ_i for $i = \lambda+2, \lambda+4, \ldots, -\lambda+2$ does not induce an isomorphism

$$\mathcal{O}_\Delta\{i-1\} \cong \mathcal{H}^{i-1}(\mathcal{P}) \xrightarrow{\gamma_i} \mathcal{H}^{i+1}(\mathcal{P})\{-2\} \cong \mathcal{O}_\Delta\{i-1\}$$

then this map must induce zero. But then $\operatorname{supp}\mathcal{H}^j(\operatorname{Cone}(\Theta(\lambda+2)))$ contains Δ for some $j \neq -\lambda, \lambda+1$. Comparing with the long exact sequence induced by (5) we find this is impossible since $\Delta \not\subset \operatorname{supp}\mathcal{H}^*(\mathcal{E}(\lambda-1) * \mathcal{F}(\lambda-1))$. Thus all the maps γ_i in the range above induce isomorphisms on the cohomology of $\mathcal{P}$.

Finally, this means that if we apply $\Theta(\lambda+2)^i$ then we get an induced map

$$\mathcal{H}^{\lambda+1}(\mathcal{P}) \xrightarrow{\gamma} \mathcal{H}^{\lambda+1+2i}(\mathcal{P})\{-2i\}$$

where $\gamma = \gamma_{-\lambda+2} \circ \cdots \circ \gamma_{\lambda+2}$, which is an isomorphism. This completes the proof. □

Corollary 4.9. *Suppose we have a geometric categorical $\mathbb{C}^\times$-equivariant $\mathfrak{sl}_2$ action. Then if $\lambda \leq 0$ we have*

$$\mathcal{F}(\lambda+1) * \mathcal{E}(\lambda+1) \cong \mathcal{E}(\lambda-1) * \mathcal{F}(\lambda-1) \oplus \mathcal{O}_\Delta \otimes_{\Bbbk} H^\star(\mathbb{P}^{-\lambda-1}).$$

while if $\lambda \geq 0$ we have

$$\mathcal{E}(\lambda-1) * \mathcal{F}(\lambda-1) \cong \mathcal{F}(\lambda+1) * \mathcal{E}(\lambda+1) \oplus \mathcal{O}_\Delta \otimes_{\Bbbk} H^\star(\mathbb{P}^{\lambda-1}).$$

Proof. Suppose $\lambda \leq 0$. Recall that $\mathcal{F}(\lambda+1) * \mathcal{E}(\lambda+1) \cong \mathcal{E}(\lambda-1) \circ \mathcal{F}(\lambda-1) \oplus \mathcal{P}$ where $\mathcal{H}^\star(\mathcal{P}) \cong \mathcal{O}_\Delta \otimes_{\Bbbk} H^\star(\mathbb{P}^{-\lambda-1})$. Consider the unit $\eta : \mathcal{O}_\Delta[-\lambda-1]\{\lambda+1\} \to \mathcal{F}(\lambda+1) * \mathcal{E}(\lambda+1)$. This is nothing but the natural inclusion of $\mathcal{O}_\Delta$ into the bottom of $\mathcal{P}$. Construct the map

$$(\Theta(\lambda+2)^{-\lambda-1} \circ \eta[2(\lambda+1)]\{2(-\lambda-1)\}) \oplus \cdots \oplus (\Theta(\lambda+2) \circ \eta[-2]\{2\}) \oplus \eta :$$
$$\bigoplus_{i=0}^{-\lambda-1} \mathcal{O}_\Delta[-\lambda-1+2i]\{\lambda+1-2i\} \to \mathcal{F}(\lambda+1) * \mathcal{E}(\lambda+1).$$

By Lemma 4.8, the composition of this map with the projection $\mathcal{F}(\lambda+1) * \mathcal{E}(\lambda+1) \to \mathcal{P}$ is an isomorphism on cohomology. This means that $\mathcal{P} \cong \mathcal{O}_\Delta \otimes_{\Bbbk} H^\star(\mathbb{P}^{-\lambda-1})$ and we are done.

The proof for $\lambda \geq 0$ is the same. □

Corollary 4.10. *Suppose one has a geometric categorical $\mathfrak{sl}_2$ action. Then if $-\lambda - a + b \geq 0$ we have*

$$\mathcal{F}^{(b)}(\lambda+2a-b) * \mathcal{E}^{(a)}(\lambda+a) \cong \bigoplus_{j \geq 0} \mathcal{E}^{(a-j)} * \mathcal{F}^{(b-j)} \otimes_{\Bbbk} H^\star(\mathbb{G}(j, -\lambda-a+b))$$

where on the right-hand side $\operatorname{End}(\mathcal{E}^{(a-j)} * \mathcal{F}^{(b-j)}) \cong \Bbbk \cdot \mathrm{I}$. *Also* $\operatorname{End}(\mathcal{E}^{(a)}(\lambda-2b+a) * \mathcal{F}^{(b)}(\lambda-b)) = \Bbbk \cdot \mathrm{I}$.

Similarly, if $\lambda - a + b \geq 0$ then we have

$$\mathcal{E}^{(b)}(\lambda-2a+b) * \mathcal{F}^{(a)}(\lambda-a) \cong \bigoplus_{j \geq 0} \mathcal{F}^{(a-j)} * \mathcal{E}^{(b-j)} \otimes_{\Bbbk} H^\star(\mathbb{G}(j, \lambda-a+b))$$

where on the right-hand side $\operatorname{End}(\mathcal{F}^{(a-j)} * \mathcal{E}^{(b-j)}) \cong \Bbbk \cdot \mathrm{I}$. *Also* $\operatorname{End}(\mathcal{F}^{(a)}(\lambda+2b-a) * \mathcal{E}^{(b)}(\lambda+b)) \cong \Bbbk \cdot \mathrm{I}$.

Proof. We will not be using this result much in this paper. See Lemma 4.2 of [CKL2] for the proof. □

5. Proof of nil affine Hecke relations

In this section we prove the following result which, together with the results in section 4, prove the main Theorem 2.7.

Theorem 5.1. *Given a geometric categorical $\mathfrak{sl}_2$ action there exist morphisms*

$$X(\lambda) : \mathcal{E}(\lambda)\langle -1 \rangle \to \mathcal{E}(\lambda)\langle 1 \rangle$$

and

$$T(\lambda) : \mathcal{E}(\lambda+1) * \mathcal{E}(\lambda-1)\langle 1 \rangle \to \mathcal{E}(\lambda+1) * \mathcal{E}(\lambda-1)\langle -1 \rangle$$

satisfying the nil affine Hecke relations

(i) $T(\lambda)^2 = 0$

(ii) $(I * T(\lambda-1)) \circ (T(\lambda+1) * I) \circ (I * T(\lambda-1)) = (T(\lambda+1) * I) \circ (I * T(\lambda-1)) \circ (T(\lambda+1) * I)$ *as morphisms* $\mathcal{E}(\lambda+2) * \mathcal{E}(\lambda) * \mathcal{E}(\lambda-2)\langle 3\rangle \to \mathcal{E}(\lambda+2) * \mathcal{E}(\lambda) * \mathcal{E}(\lambda-2)\langle -3\rangle$.

(iii) $(X(\lambda+1) * I) \circ T(\lambda) - T(\lambda) \circ (I * X(\lambda-1)) = I = -(I * X(\lambda-1)) \circ T(\lambda) + T(\lambda) \circ (X(\lambda+1) * I)$ *as morphisms* $\mathcal{E}(\lambda+1) * \mathcal{E}(\lambda-1) \to \mathcal{E}(\lambda+1) * \mathcal{E}(\lambda-1)$.

The freedom in choosing such X*s and* T*s is parametrized by*

$$V(-1)^{tr} \times V(-2)^{tr} \times \Bbbk^\times \cong V(1)^{tr} \times V(2)^{tr} \times \Bbbk^\times$$

where $V(\lambda)^{tr} \subset \mathrm{Hom}(\mathcal{O}_\Delta\langle -1\rangle, \mathcal{O}_\Delta\langle 1\rangle)$ *denotes the linear subspace of transient maps defined below.*

In this section, we use the notation $\langle k\rangle$ for $[k]\{-k\}$.

5.1. **Construction and proof of Relation (iii).** We first construct Xs and Ts satisfying nil affine Hecke relation (iii).

5.1.1. $\theta(\lambda)$ *and Transient Maps.* The first step is to better understand maps $\mathcal{E}(\lambda)\langle -1\rangle \to \mathcal{E}(\lambda)\langle 1\rangle$. We will write $\Delta(\lambda)$ for the diagonal inside $Y(\lambda) \times Y(\lambda)$ when we want to emphasize where the diagonal lives.

Without loss of generality we can assume $\lambda \le 0$. Then

$$\mathrm{Hom}(\mathcal{E}(\lambda), \mathcal{E}(\lambda)\langle 2\rangle) \cong \mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{E}(\lambda)_R * \mathcal{E}(\lambda)\langle 2\rangle) \cong \mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{F}(\lambda) * \mathcal{E}(\lambda)\langle \lambda+2\rangle).$$

Now $\mathcal{F}(\lambda) * \mathcal{E}(\lambda) \cong \mathcal{E}(\lambda-2) * \mathcal{F}(\lambda-2) \oplus \mathcal{O}_\Delta \otimes_\Bbbk H^\star(\mathbb{P}^{-\lambda})$ and we have that

$$\begin{aligned}\mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{E}(\lambda-2) * \mathcal{F}(\lambda-2)\langle \lambda+2\rangle) &\cong \mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{F}(\lambda-2)_R * \mathcal{F}(\lambda-2)\langle 2\lambda\rangle)\\ &\cong \mathrm{Hom}(\mathcal{F}(\lambda-2), \mathcal{F}(\lambda-2)\langle 2\lambda\rangle).\end{aligned}$$

is zero if $\lambda \le -1$. So if $\lambda \le -1$ we get

$$\begin{aligned}\mathrm{Hom}(\mathcal{E}(\lambda), \mathcal{E}(\lambda)\langle 2\rangle) &\cong \mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{O}_\Delta \otimes_\Bbbk H^\star(\mathbb{P}^{-\lambda})\langle \lambda+1\rangle)\\ &\cong \mathrm{Hom}(\mathcal{O}_{\Delta(\lambda-1)}, \mathcal{O}_{\Delta(\lambda-1)}) \oplus \mathrm{Hom}(\mathcal{O}_{\Delta(\lambda-1)}, \mathcal{O}_{\Delta(\lambda-1)}\langle 2\rangle).\end{aligned}$$

Meanwhile, if $\lambda = 0$ we get

$$\mathrm{Hom}(\mathcal{E}(\lambda), \mathcal{E}(\lambda)\langle 2\rangle) \cong \mathrm{Hom}(\mathcal{F}(\lambda-2), \mathcal{F}(\lambda-2)) \oplus \mathrm{Hom}(\mathcal{O}_{\Delta(\lambda-1)}, \mathcal{O}_{\Delta(\lambda-1)}\langle 2\rangle).$$

In both cases we have the maps indexed by $\mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{O}_\Delta\langle 2\rangle)$. If we examine the adjunction calculation above it is easy to see that these maps correspond to those of the form

$$\mathcal{E}(\lambda) * (\mathcal{O}_{\Delta(\lambda-1)}\langle -1\rangle \to \mathcal{O}_{\Delta(\lambda-1)}\langle 1\rangle).$$

On the other hand, there is also the map

$$(\mathcal{O}_{\Delta(\lambda+1)}\langle -1\rangle \xrightarrow{\theta(\lambda+1)} \mathcal{O}_{\Delta(\lambda+1)}\langle 1\rangle) * \mathcal{E}(\lambda).$$

This map cannot be of the form above because if it were then

$$\begin{aligned}&\mathcal{E}(\lambda+2) * (\mathcal{O}_{\Delta(\lambda+1)}\langle -1\rangle \xrightarrow{\theta(\lambda+1)} \mathcal{O}_{\Delta(\lambda+1)}\langle 1\rangle) * \mathcal{E}(\lambda)\\ \cong\ &\mathcal{E}(\lambda+2) * \mathcal{E}(\lambda) * (\mathcal{O}_{\Delta(\lambda-1)}\langle -1\rangle \to \mathcal{O}_{\Delta(\lambda-1)}\langle 1\rangle)\\ \cong\ &\mathcal{E}^{(2)}(\lambda+1) * (\mathcal{O}_{\Delta(\lambda-1)}\langle -1\rangle \to \mathcal{O}_{\Delta(\lambda-1)}\langle 1\rangle) \otimes_\Bbbk H^\star(\mathbb{P}^1)\end{aligned}$$

induces the zero map on the cohomology $\mathcal{E}^{(2)}(\lambda+1)\langle -1\rangle \oplus \mathcal{E}^{(2)}(\lambda+1)\langle 1\rangle$ of $\mathcal{E}(\lambda+2) * \mathcal{E}(\lambda)$ (note that the map itself is not zero). This contradicts Lemma 4.5 in the case $i = 1$.

We conclude that any map $\mathcal{E}(\lambda)\langle -1\rangle \to \mathcal{E}(\lambda)\langle 1\rangle$ is of the form $a\theta(\lambda+1) + \phi$ where $\phi : \mathcal{O}_{\Delta(\lambda-1)}\langle -1\rangle \to \mathcal{O}_{\Delta(\lambda-1)}\langle 1\rangle$ is arbitrary. Notationally we will write this map as $(a\theta(\lambda+1), \phi)$ to remind ourselves that $\theta(\lambda+1)$ acts on the left of $\mathcal{E}(\lambda)$ while ϕ acts on the right.

There is one last piece of nice structure here worth exploiting. Take any

$$\phi \in \mathrm{Hom}(\mathcal{O}_{\Delta(\lambda-1)}\langle -1\rangle, \mathcal{O}_{\Delta(\lambda-1)}\langle 1\rangle).$$

Then by the argument above

$$(\mathcal{O}_{\Delta(\lambda-1)}\langle -1\rangle \xrightarrow{\phi} \mathcal{O}_{\Delta(\lambda-1)}\langle 1\rangle) * \mathcal{E}(\lambda-2)$$

must be equal to a map $(a'\theta(\lambda-1), \phi')$ for some $a' \in \Bbbk$ and $\phi' : \mathcal{O}_{\Delta(\lambda-3)}\langle -1\rangle \to \mathcal{O}_{\Delta(\lambda-3)}\langle 1\rangle$. This means that there exists a distinguished linear subspace

$$V(\lambda-1)^{tr} \subset \mathrm{Hom}(\mathcal{O}_{\Delta(\lambda-1)}\langle -1\rangle, \mathcal{O}_{\Delta(\lambda-1)}\langle 1\rangle)$$

consisting of those ϕ which induce $a' = 0$. We will call such an element $\phi \in V(\lambda-1)^{tr}$ a *t*ransient map.

We can define transient maps $V(\lambda)^{tr} \subset \mathrm{Hom}(\mathcal{O}_{\Delta(\lambda)}\langle -1\rangle, \mathcal{O}_{\Delta(\lambda)}\langle 1\rangle)$ for every $Y(\lambda)$ by using $\mathcal{E}(\lambda-1)$ if $\lambda \le 0$ and $\mathcal{F}(\lambda+1)$ if $\lambda \ge 0$. There is a small conflict when $\lambda = 0$ since there are two ways of defining transient maps in that case. Fortunately the two definitions agree (see Proposition 5.3 below).

Notice that $V(\lambda)^{tr} \subset \mathrm{Hom}(\mathcal{O}_{\Delta(\lambda)}\langle -1\rangle, \mathcal{O}_{\Delta(\lambda)}\langle 1\rangle)$ is a codimension one linear subspace (the only exception is at the extremes where $\lambda = \pm N$ in which case every map is transient). If ϕ is transient then we can "slide" it from the $Y(\lambda)$ slot to the $Y(\lambda-2)$ slot if $\lambda \le 0$ ($Y(\lambda+2)$ slot if $\lambda \ge 0$) to obtain some new ϕ'. As Proposition 5.3 below shows ϕ' will again be transient and so we can repeat the process. This is why we call them transient maps. To summarize:

Proposition 5.2. *Every map $\mathcal{E}(\lambda)\langle -1\rangle \to \mathcal{E}(\lambda)\langle 1\rangle$ is of the form $(a\theta(\lambda+1), b\theta(\lambda-1)+\phi)$ if $\lambda \le 0$ and $(a\theta(\lambda+1)+\phi, b\theta(\lambda-1))$ if $\lambda \ge 0$ where $a, b \in \Bbbk$ and ϕ is transient. Taking adjoints we obtain the analogous claim for maps $\mathcal{F}(\lambda)\langle -1\rangle \to \mathcal{F}(\lambda)\langle 1\rangle$.*

Proposition 5.3. *Transient maps are well defined and come equipped with natural maps $V(\lambda)^{tr} \to V(\lambda-2)^{tr}$ if $\lambda \le 0$ and $V(\lambda)^{tr} \to V(\lambda+2)^{tr}$ if $\lambda \ge 0$. We also have isomorphisms $V(-1)^{tr} \cong V(1)^{tr}$ and $V(-2)^{tr} \cong V(2)^{tr}$. For $\lambda \ne \pm N$, the quotient $\mathrm{Hom}(\mathcal{O}_{\Delta(\lambda)}\langle -1\rangle, \mathcal{O}_{\Delta(\lambda)}\langle 1\rangle)/V(\lambda)^{tr}$, which is one-dimensional, is spanned by $[\theta(\lambda)\rangle$. If $\lambda = \pm N$ this quotient is zero.*

Proof. To obtain the natural maps note that if $\phi \in V(\lambda)^{tr}$ (where $\lambda \le 0$) then by definition

$$(\phi, 0) : \mathcal{E}(\lambda-1)\langle -1\rangle \to \mathcal{E}(\lambda-1)\langle 1\rangle$$

is isomorphic to $(0, \phi')$ for a unique $\phi' \in V(\lambda-2)$. What we need to check is that ϕ' is transient.

To this this we consider

$$(\phi, 0, 0) = (0, \phi', 0) : \mathcal{E}(\lambda-1) * \mathcal{E}(\lambda-3)\langle -1\rangle \to \mathcal{E}(\lambda-1) * \mathcal{E}(\lambda-3)\langle 1\rangle.$$

We can do this unless $\lambda - 2 = -N$ in which case ϕ' is automatically transient since all such maps are transient. Then $(0, \phi', 0) = (0, a\theta(\lambda-2), \phi'')$ for some

$$a \in \Bbbk \text{ and } \phi'' \in \mathrm{Hom}(\mathcal{O}_{\Delta(\lambda-4)}\langle -1\rangle, \mathcal{O}_{\Delta(\lambda-4)}\langle 1\rangle).$$

We need to show that $a = 0$. To do this we consider the cone

$$\mathrm{Cone}(\mathcal{E}(\lambda-1) * \mathcal{E}(\lambda-3)\langle -1\rangle \xrightarrow{(0, a\theta(\lambda-2), \phi'')} \mathcal{E}(\lambda-1) * \mathcal{E}(\lambda-3)\langle 1\rangle)$$

If $a \ne 0$ then the induced map on cohomology is an isomorphism in homological degree zero and so the cone has non-zero cohomology only in degrees -2 and 1. But this cone is the same $\mathrm{Cone}(\phi, 0, 0)$ which does not induce an isomorphism in degree zero and hence has non-zero cohomology in homological degrees $-2, -1, 0, 1$. Thus $a = 0$ and ϕ' is transient.

The case of $\lambda \ge 0$ follows similarly.

That $V(-1)^{tr} \cong V(1)^{tr}$ follows from the fact that we have natural maps $V(-1)^{tr} \to V(1)^{tr} \to V(-1)^{tr}$ whose composition is the identity. To see that the composition is the identity observe that a map

$$\mathcal{E}(0)\langle -1\rangle \to \mathcal{E}(0)\langle 1\rangle$$

is of the form $(a\theta(1), b\theta(-1) + \phi)$ for a *unique* transient ϕ.

The isomorphism $V(-2)^{tr} \cong V(2)^{tr}$ follows similarly by looking at maps $\mathcal{E}^{(2)}(0)\langle -1 \rangle \to \mathcal{E}^{(2)}(0)\langle 1 \rangle$ and using Proposition 5.7 which states that any such map is of the form $(a\theta(2), b\theta(-2)+\phi)$ for a unique transient ϕ. □

5.1.2. *Defining the Ts and Xs modulo transient maps.* As a first step we will define the $X(\lambda)$s up to transients. Working modulo transients is more convenient since (for $\lambda \neq \pm N$)

$$\mathrm{Hom}(\mathcal{E}(\lambda), \mathcal{E}(\lambda)\langle 2 \rangle) \text{ modulo transients } \cong \{(a\theta(\lambda+1), b\theta(\lambda-1)) : a, b \in \Bbbk\}$$

is two-dimensional spanned by $(0, \theta(\lambda-1))$ and $(\theta(\lambda+1), 0)$. Thus to determine $X(\lambda)$ modulo transients we only need to choose $a(\lambda), b(\lambda) \in \Bbbk^2$ and define $X(\lambda) := (a(\lambda)\theta(\lambda+1), b(\lambda)\theta(\lambda-1))$.

We begin by fixing an isomorphism

$$\mathcal{E}(\lambda+1) * \mathcal{E}(\lambda-1) \xrightarrow{\sim} \mathcal{E}^{(2)}(\lambda)\langle -1 \rangle \oplus \mathcal{E}^{(2)}(\lambda)\langle 1 \rangle. \tag{6}$$

This isomorphism is not unique since we can compose it with elements of

$$\mathrm{Aut}(\mathcal{E}^{(2)}(\lambda)\langle -1 \rangle \oplus \mathcal{E}^{(2)}(\lambda)\langle 1 \rangle) \cong \left\{ \begin{pmatrix} a \cdot \mathrm{I} & 0 \\ \alpha & b \cdot \mathrm{I} \end{pmatrix} : a, b \in \Bbbk^\times, \alpha \in \mathrm{Hom}(\mathcal{E}^{(2)}(\lambda)\langle -1 \rangle, \mathcal{E}^{(2)}(\lambda)\langle 1 \rangle) \right\}$$

(here we use that $\mathrm{Ext}^i(\mathcal{E}^{(k)}(\lambda), \mathcal{E}^{(k)}(\lambda)\{j\}) = 0$ for $i < 0$ and any $j \in \mathbb{Z}$ while $\mathrm{End}(\mathcal{E}^{(k)}(\lambda)) \cong \Bbbk \cdot \mathrm{I}$).

Using this isomorphism we can write

$$\mathrm{I} * X(\lambda-1) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{E}^{(2)}(\lambda)\langle -2 \rangle \\ \mathcal{E}^{(2)}(\lambda) \end{pmatrix} \to \begin{pmatrix} \mathcal{E}^{(2)}(\lambda) \\ \mathcal{E}^{(2)}(\lambda)\langle 2 \rangle \end{pmatrix}$$

where $A, D \in \mathrm{Hom}(\mathcal{E}^{(2)}(\lambda), \mathcal{E}^{(2)}(\lambda)\langle 2 \rangle)$, $B \in \mathrm{End}(\mathcal{E}^{(2)}(\lambda))$ and $C \in \mathrm{Hom}(\mathcal{E}^{(2)}(\lambda), \mathcal{E}^{(2)}(\lambda)\langle 4 \rangle)$. Similarly, we have

$$X(\lambda+1) * I = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} : \begin{pmatrix} \mathcal{E}^{(2)}(\lambda)\langle -2 \rangle \\ \mathcal{E}^{(2)}(\lambda) \end{pmatrix} \to \begin{pmatrix} \mathcal{E}^{(2)}(\lambda) \\ \mathcal{E}^{(2)}(\lambda)\langle 2 \rangle \end{pmatrix}.$$

Note that although these two matrices are defined only up to conjugation their traces $A + D$ and $A' + D'$ as well as B and B' are invariant under conjugation.

Lemma 5.4. *If $a(\lambda-1) \neq 0$ then B is a non-zero multiple of I. Similarly, if $b(\lambda+1) \neq 0$ then B' is a non-zero multiple of I.*

Proof. Since $I * X(\lambda-1) = (0, a(\lambda-1)\theta(\lambda), b(\lambda-1)\theta(\lambda-2))$ we know that (at the level of cohomology)

$$\begin{aligned} \mathcal{H}(\mathrm{Cone}(I * X(\lambda-1))) &\cong \mathcal{H}(\mathrm{Cone}(0, a(\lambda-1)\theta(\lambda), 0) \\ &\cong \mathcal{E}^{(2)}(\lambda) \otimes_{\Bbbk} (\Bbbk[-1]\{1\} \oplus \Bbbk[2]\{-3\}) \end{aligned}$$

since $a(\lambda-1) \neq 0$. But the long exact sequence in cohomology induced by $I * X(\lambda-1)$ looks like

$$\cdots \to 0 \to \mathcal{E}^{(2)}(\lambda) \xrightarrow{B} \mathcal{E}^{(2)}(\lambda) \to 0 \to \dots$$

so that B has to be an isomorphism. Since $\mathrm{End}(\mathcal{E}^{(2)}(\lambda)) = \Bbbk \cdot \mathrm{I}$ the result follows.

The result for B' follows similarly. □

At this point we define

$$T(\lambda) := \begin{pmatrix} 0 & 0 \\ -B^{-1} & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{E}^{(2)}(\lambda) \\ \mathcal{E}^{(2)}(\lambda)\langle -2 \rangle \end{pmatrix} \to \begin{pmatrix} \mathcal{E}^{(2)}(\lambda)\langle 2 \rangle \\ \mathcal{E}^{(2)}(\lambda) \end{pmatrix}. \tag{7}$$

Notice that this map is invariant under conjugation and hence does not depend on our choice of isomorphism (6). In this notation it is now easy to check that nil affine Hecke relation (iii) is equivalent to the conditions

$$B + B' = 0 = C + C' \text{ and } A = D' \text{ and } A' = D.$$

Remark 5.5. A second way to characterize nil affine Hecke relation (iii) is by the conditions that $I * X(\lambda-1) + X(\lambda+1) * I$ is a multiple of the identity and

$$\mathrm{Trace}(I * X(\lambda-1)) = \mathrm{Trace}(X(\lambda+1) * I) : \mathcal{E}^{(2)}(\lambda)\langle -1\rangle \to \mathcal{E}^{(2)}(\lambda)\langle 1\rangle.$$

This second condition can also be replaced by asking that

$$X(\lambda+1) * X(\lambda-1) : \mathcal{E}(\lambda+1) * \mathcal{E}(\lambda-1)\langle -2\rangle \to \mathcal{E}(\lambda+1) * \mathcal{E}(\lambda-1)\langle 2\rangle$$

is diagonal.

We will now recursively define the Xs. As a first step we let $b(\lambda+1) = -a(\lambda-1)$. Then we begin with the smallest weight by first defining

$$X(-N+1) := (\theta(-N+2), 0) : \mathcal{E}(-N+1)\langle -1\rangle \to \mathcal{E}(-N+1)\langle 1\rangle.$$

Notice that on $Y(-N)$ all maps $\mathcal{O}_\Delta\langle -1\rangle \to \mathcal{O}_\Delta\langle 1\rangle$ are transient so the only choice we have is which (non-zero) multiple of $\theta(-N+2)$ we should take. Clearly the space of such choices is parametrized by $\Bbbk^\times$.

Now suppose by induction that we have defined $X(-N+1), \ldots, X(\lambda-1), X(\lambda+1) = (a(\lambda+1)\theta(\lambda+2), -a(\lambda-1)\theta(\lambda))$ such that nil affine Hecke relation (iii) holds for every pair up to $\mathcal{E}(\lambda+1) * \mathcal{E}(\lambda-1)$ and such that all the a's are non-zero. This means

$$X(\lambda+3) := (a(\lambda+3)\theta(\lambda+4), -a(\lambda+1)\theta(\lambda+2)) : \mathcal{E}(\lambda+3)\langle -1\rangle \to \mathcal{E}(\lambda+3)\langle 1\rangle$$

where $a(\lambda+3)$ remains to be determined.

Let's fix again some isomorphism

$$\mathcal{E}(\lambda+3) * \mathcal{E}(\lambda+1) \xrightarrow{\sim} \mathcal{E}^{(2)}(\lambda+2)\langle -1\rangle \oplus \mathcal{E}^{(2)}(\lambda+2)\langle 1\rangle \tag{8}$$

under which we have the identifications

$$I * X(\lambda+1) = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix} \text{ and } X(\lambda+3) * I = \begin{pmatrix} \hat{A}' & \hat{B}' \\ \hat{C}' & \hat{D}' \end{pmatrix}.$$

Lemma 5.6. *We have* $\hat{B} + \hat{B}' = 0 = \hat{C} + \hat{C}'$ *while*

$$\hat{A} + \hat{A}' = \hat{D} + \hat{D}' = (a(\lambda+3)\theta(\lambda+4), b(\lambda+1)\theta(\lambda)) : \mathcal{E}^{(2)}(\lambda+2)\langle -1\rangle \to \mathcal{E}^{(2)}(\lambda+2)\langle 1\rangle.$$

Proof. We have

$$I * X(\lambda+1) = (0, a(\lambda+1)\theta(\lambda+2), -a(\lambda-1)\theta(\lambda))$$

and

$$X(\lambda+3) * I = (a(\lambda+3)\theta(\lambda+4), -a(\lambda+1)\theta(\lambda+2), 0)).$$

Hence

$$I * X(\lambda+1) + X(\lambda+3) * I = (a(\lambda+3)\theta(\lambda+4), 0, -a(\lambda-1)\theta(\lambda))$$

and the result follows. □

It remains to show that we can choose $a(\lambda+3) \neq 0$ such that $\hat{A} = \hat{D}'$ and $\hat{A}' = \hat{D}$. To do this we first need to understand the possible maps $\mathcal{E}^{(2)}(\lambda)\langle -1\rangle \to \mathcal{E}^{(2)}(\lambda)\langle 1\rangle$.

Proposition 5.7. *Every map* $\mathcal{E}^{(2)}(\lambda)\langle -1\rangle \to \mathcal{E}^{(2)}(\lambda)\langle 1\rangle$ *is of the form* $(a\theta(\lambda+2), b\theta(\lambda-2) + \phi)$ *if* $\lambda \leq 0$ *and* $(a\theta(\lambda+2) + \phi, b\theta(\lambda-2))$ *if* $\lambda \geq 0$ *where* $a, b \in \Bbbk$ *and* ϕ *is transient.*

Proof. This result is analogous to Proposition 5.2 and the proof is very similar.

Suppose $\lambda \leq 0$. Then

$$\begin{aligned}\mathrm{Hom}(\mathcal{E}^{(2)}(\lambda), \mathcal{E}^{(2)}(\lambda)\langle 2\rangle) &\cong \mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{E}^{(2)}(\lambda)_R * \mathcal{E}^{(2)}(\lambda)\langle 2\rangle) \\ &\cong \mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{F}^{(2)}(\lambda) * \mathcal{E}^{(2)}(\lambda)\langle 2\lambda+2\rangle).\end{aligned}$$

Now by Corollary 4.10

$$\mathcal{F}^{(2)}(\lambda) * \mathcal{E}^{(2)}(\lambda) \cong \mathcal{E}^{(2)}(\lambda-4) * \mathcal{F}^{(2)}(\lambda-4) \oplus \mathcal{E}(\lambda-3) * \mathcal{F}(\lambda-3) \otimes_{\Bbbk} H^\star(\mathbb{G}(1,-\lambda+2)) \oplus \mathcal{O}_\Delta \otimes_{\Bbbk} H^\star(\mathbb{G}(2,-\lambda+2)).$$

Now

$$\begin{aligned}\mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{E}^{(2)}(\lambda-4) * \mathcal{F}^{(2)}(\lambda-4)\langle 2\lambda+2\rangle) &\cong \mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{F}^{(2)}(\lambda-4)_R * \mathcal{F}^{(2)}(\lambda-4)\langle 4\lambda-6\rangle)) \\ &\cong \mathrm{Hom}(\mathcal{F}^{(2)}(\lambda-4), \mathcal{F}^{(2)}(\lambda-4)\langle 4\lambda-6\rangle)\end{aligned}$$

is zero for $\lambda \le 0$. Also

$$\begin{aligned}\mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{E}(\lambda-3) * \mathcal{F}(\lambda-3)) &\cong \mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{F}(\lambda-3)_R * \mathcal{F}(\lambda-3)\langle \lambda-3\rangle) \\ &\cong \mathrm{Hom}(\mathcal{F}(\lambda-3), \mathcal{F}(\lambda-3)\langle \lambda-3\rangle)\end{aligned}$$

so that

$$\mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{E}(\lambda-3)*\mathcal{F}(\lambda-3)\otimes_{\Bbbk} H^\star(\mathbb{G}(1,-\lambda+2)\langle 2\lambda+2\rangle) \cong \mathrm{Hom}(\mathcal{F}(\lambda-3), \mathcal{F}(\lambda-3)\otimes_{\Bbbk} H^\star(\mathbb{P}^{-\lambda+1})\langle 3\lambda-1\rangle)$$

is zero if $\lambda < 0$ and isomorphic to $\mathrm{Hom}(\mathcal{F}(-3), \mathcal{F}(-3) \otimes_{\Bbbk} H^\star(\mathbb{P}^1)\langle 1\rangle) \cong \Bbbk$ if $\lambda = 0$.

Finally, notice that $H^\star(\mathbb{G}(2,-\lambda+2))$ is supported in homological degrees $\ge 2\lambda$ and one-dimensional in lowest homological degrees 2λ and $2\lambda+2$ (remember $\lambda \le 0$). It follows that if $\lambda < 0$

$$\mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{O}_\Delta \otimes_{\Bbbk} H^\star(\mathbb{G}(2,-\lambda+2)\langle 2\lambda+2\rangle) \cong \mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \oplus \mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{O}_\Delta\langle 2\rangle)$$

while if $\lambda = 0$ we get $\mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{O}_\Delta \otimes_{\Bbbk} H^\star(\mathbb{G}(2,2)\langle 2\rangle) \cong \mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{O}_\Delta\langle 2\rangle)$.

So if $\lambda \le 0$ the space of maps $\mathrm{Hom}(\mathcal{E}^{(2)}(\lambda), \mathcal{E}^{(2)}(\lambda)\langle 2\rangle)$ is spanned by maps of the form $(0, b\theta(\lambda-2)+\phi)$ (corresponding to the factor $\mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{O}_\Delta\langle 2\rangle)$ in the calculation above) and one more map (corresponding to the factor $\mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \cong \Bbbk$). Since $(\theta(\lambda+2), 0)$ is linearly independent to the maps above (as a corollary of Lemma 4.5) we can use it as the extra map.

The proof for $\lambda \ge 0$ is analogous. □

Proposition 5.8. *There exists a unique $a(\lambda+3) \ne 0$ so that $\hat{A} = \hat{D}'$ and $\hat{A}' = \hat{D}$ in the notation above.*

Proof. Choose $a(\lambda+3)$ arbitrarily and consider the map

$$\begin{aligned}X(\lambda+3) * X(\lambda+1) * X(\lambda-1) \quad &: \quad \mathcal{E}(\lambda+3) * \mathcal{E}(\lambda+1) * \mathcal{E}(\lambda-1)\langle -3\rangle \\ &\to \quad \mathcal{E}(\lambda+3) * \mathcal{E}(\lambda+1) * \mathcal{E}(\lambda-1)\langle 3\rangle\end{aligned}$$

at the level of cohomology. Recall that

$$\mathcal{E}(\lambda+3) * \mathcal{E}(\lambda+1) * \mathcal{E}(\lambda-1) \cong \mathcal{E}^{(3)}(\lambda+1) \otimes_{\Bbbk} H^\star(\mathbb{F}l_3).$$

Now

$$X(\lambda+1) * X(\lambda-1) = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} AD-BC & 0 \\ 0 & AD-BC \end{pmatrix}$$

since by induction we have nil affine Hecke relation (iii) and so $B + B' = 0 = C + C'$ and $A = D'$ and $A' = D$. This means that

$$X(\lambda+3) * X(\lambda+1) * X(\lambda-1) = X(\lambda+3) * (AD-BC) \otimes_{\Bbbk} H^\star(\mathbb{P}^1)$$

where

$$X(\lambda+3) * (AD-BC) : \mathcal{E}(\lambda+3) * \mathcal{E}^{(2)}(\lambda)\langle -3\rangle \to \mathcal{E}(\lambda+3) * \mathcal{E}^{(2)}(\lambda)\langle 3\rangle.$$

Since $\mathcal{E}(\lambda+3) * \mathcal{E}^{(2)}(\lambda) \cong \mathcal{E}^{(3)} \otimes_{\Bbbk} H^\star(\mathbb{P}^2)$ such a map is automatically zero at the level of cohomology (regardless of our choice of $X(\lambda+3)$). Thus $X(\lambda+3) * X(\lambda+1) * X(\lambda-1)$ induces zero at the level of cohomology.

On the other hand we can consider

$$X(\lambda+3) * X(\lambda+1) = \begin{pmatrix} \hat{A}' & \hat{B}' \\ \hat{C}' & \hat{D}' \end{pmatrix} \cdot \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix} = \begin{pmatrix} * & \hat{B}(\hat{A}' - \hat{D}) \\ * & * \end{pmatrix}$$

where we use that $\hat{B} + \hat{B}' = 0$. Each entry marked $*$ (whose precise value we do not care about) has homological degree four or six. So by degree considerations each entry $*$ induces zero on the cohomology of

$$\mathcal{E}(\lambda+3) * \mathcal{E}(\lambda+1) \cong \mathcal{E}^{(2)}(\lambda+2) \otimes_{\Bbbk} H^\star(\mathbb{P}^1).$$

Hence

$$\hat{B}(\hat{A}' - \hat{D}) * X(\lambda-1) : \mathcal{E}^{(2)}(\lambda+2) * \mathcal{E}(\lambda-1)\langle -2 \rangle \to \mathcal{E}^{(2)}(\lambda+2) * \mathcal{E}(\lambda-1)\langle 2 \rangle$$

must also induce zero at the level of cohomology. Notice that $a(\lambda+1) \neq 0$ by induction so that $\hat{B} \neq 0$ by Lemma 5.4 and hence $(\hat{A}' - \hat{D}) * X(\lambda-1)$ must induce zero.

By proposition 5.7 we know that (modulo transient maps) $(\hat{A}' - \hat{D}) = (u\theta(\lambda+4), v\theta(\lambda))$ for some $u, v \in \Bbbk$. Also $X(\lambda-1) = (a(\lambda-1)\theta(\lambda), -a(\lambda-3)\theta(\lambda-2))$ where $a(\lambda-1) \neq 0$. Thus if $v \neq 0$ then by Lemma 4.5 both

$$(\hat{A}' - \hat{D}) * I \text{ and } I * X(\lambda-1) : \mathcal{E}^{(3)}(\lambda+1)\langle -1 \rangle \otimes_{\Bbbk} H^\star(\mathbb{P}^2) \to \mathcal{E}^{(3)}(\lambda+1)\langle 1 \rangle \otimes_{\Bbbk} H^\star(\mathbb{P}^2)$$

induce an isomorphism on the two copies of $\mathcal{E}^{(3)}(\lambda+1)$ in homological degrees -1 and 1. Consequently, the composition

$$((\hat{A}' - \hat{D}) * I) \circ (I * X(\lambda-1)) = (\hat{A}' - \hat{D}) * X(\lambda-1)$$

would induce an isomorphism on one copy of $\mathcal{E}^{(3)}(\lambda+1)$. But we showed above this is not the case so $v = 0$.

Now notice that the map

$$(\theta(\lambda+4), 0, 0) : \mathcal{E}(\lambda+3) * \mathcal{E}(\lambda+1)\langle -1 \rangle \to \mathcal{E}(\lambda+3) * \mathcal{E}(\lambda+1)\langle 1 \rangle$$

corresponds to $\begin{pmatrix} \theta(\lambda+4) & 0 \\ 0 & \theta(\lambda+4) \end{pmatrix}$ under the isomorphism (8). Thus for $c \in \Bbbk$ we get

$$(X(\lambda+3) + (c\theta(\lambda+4), 0)) * X(\lambda+1) = \begin{pmatrix} * & \hat{B}(u\theta(\lambda+4), 0) \\ * & * \end{pmatrix} + \begin{pmatrix} * & \hat{B}c\theta(\lambda+4) \\ * & * \end{pmatrix}.$$

So if we take $c = -u$ and replace $X(\lambda+3)$ by $X(\lambda+3) + (c\theta(\lambda+4), 0)$ then we get that $\hat{A}' = \hat{D}$. Since $\hat{A} + \hat{A}' = \hat{A} + \hat{D}'$ we also get $\hat{A} = \hat{D}'$ and hence there exists a unique $a(\lambda+3)$ as required.

The only thing left is to prove that $a(\lambda+3) \neq 0$. To do this consider

$$\mathcal{E}(\lambda+5) * \mathcal{E}(\lambda+3) * \mathcal{E}(\lambda+1).$$

Note that if $\mathcal{E}(\lambda+5) = 0$ (i.e. $\lambda+5 \geq N$ so we are past the highest weight) then $Y(\lambda+4) = Y(N)$ and hence $\theta(\lambda+4) = 0$ so there is nothing to prove.

By construction we know that $X(\lambda+3) * X(\lambda+1)$ is diagonal. From the argument above we know this means that

$$X(\lambda+5) * X(\lambda+3) = \begin{pmatrix} * & (u'\theta(\lambda+6), 0) \\ * & * \end{pmatrix} \tag{9}$$

for any $X(\lambda+5)$ we like. But if $a(\lambda+3) = 0$ then

$$I * X(\lambda+3) = (0, 0, -a(\lambda+1)\theta(\lambda+2)) = \begin{pmatrix} (0, -a(\lambda+1)\theta(\lambda+2)) & 0 \\ 0 & (0, -a(\lambda+1)\theta(\lambda+2)) \end{pmatrix}$$

while we can take

$$X(\lambda+5) * I = (0, \theta(\lambda+4), 0) = \begin{pmatrix} * & \beta \\ * & * \end{pmatrix}$$

where $\beta \in \Bbbk^\times$. Then

$$X(\lambda+5) * X(\lambda+3) = \begin{pmatrix} * & \beta(0, -a(\lambda+1)\theta(\lambda+2)) \\ * & * \end{pmatrix}$$

contradicting equation (9). Thus $a(\lambda+3) \neq 0$ and we are done. □

Thus repeatedly using Proposition 5.8 we find that:

Corollary 5.9. *There exist non-zero as such that the Xs defined by*

$$X(\lambda+1) := (a(\lambda+1)\theta(\lambda+2), -a(\lambda-1)\theta(\lambda))$$

together with the Ts defined by equation (7) satisfy nil affine Hecke relation (iii) (modulo transients).

5.1.3. *Defining the Xs on the nose.* At this point we can choose our Xs and Ts so that they satisfy nil affine Hecke relation (iii) modulo transients. We will now readjust these Xs by appropriate transients so that relation (iii) holds on the nose.

If N is odd we start with $X(0)$ which we keep unchanged. Now any map

$$(\phi(1), \phi(-1), \phi(-3)) : \mathcal{E}(0) * \mathcal{E}(-2)\langle -1 \rangle \to \mathcal{E}(0) * \mathcal{E}(-2)\langle 1 \rangle$$

where the ϕ is a transient map is equivalent to a map $(0, 0, \phi)$ since we can slide over transient maps. So under the isomorphism

$$\mathcal{E}(0) * \mathcal{E}(-2) \xrightarrow{\sim} \mathcal{E}^{(2)}(-1)\langle -1 \rangle \oplus \mathcal{E}^{(2)}(-1)\langle 1 \rangle$$

any combination of transient maps is of the form $\begin{pmatrix} \phi & 0 \\ 0 & \phi \end{pmatrix}$.

Since $X(0)$ and $X(-2)$ satisfy nil affine Hecke relation (iii) modulo transients we conclude that

$$X(0) * I = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ and } I * X(-2) = \begin{pmatrix} D+\phi & -B \\ -C & A+\phi \end{pmatrix}$$

for some transient map $\phi : \mathcal{O}_{\Delta(-3)}\langle -1 \rangle \to \mathcal{O}_{\Delta(-3)}\langle 1 \rangle$.

So if we replace $X(-2) = (a(-2)\theta(-1), b(-2)\theta(-3))$ by

$$X(-2) := (a(-2)\theta(-1), b(-2)\theta(-3) - \phi)$$

then we get nil affine Hecke relation (iii) on the nose. Notice that the ϕ by which we had to change $X(-2)$ is completely determined. Now we can repeat with $X(-4), X(-6), \dots X(-N)$ and then similarly with $X(2), X(4), \dots, X(N)$. The overall freedom we had in redefining the Xs comes from being able to choose $X(0)$ arbitrarily. This choice is parametrized by $V(1)^{tr} \cong V(-1)^{tr}$.

If N is even we do the same thing except starting with $X(-1)$. We then recursively redefine $X(-3), \dots, X(-N)$ as above followed by $X(1), \dots, X(N)$. This time the freedom we have in redefining the Xs comes from being able to choose $X(-1)$ arbitrarily. This choice is parametrized by $V(-2)^{tr}$. Notice that by symmetry we could have started with $X(1)$ and then the freedom would have been parametrized by $V(2)^{tr}$ but by Proposition 5.3 $V(-2)^{tr} \cong V(2)^{tr}$.

This completes the proof of nil affine Hecke relation (iii) in Theorem 5.1 as well as the proof regarding the freedom we have in choosing the Xs and Ts.

5.2. **Proof of nil affine Hecke Relations (i) and (ii).** Nil Hecke relations (i) and (ii) now follow fairly easily from relation (iii).

Relation (i) is immediate. Notice that

$$T(\lambda)^2 : \mathcal{E}^{(2)}(\lambda) \otimes_\Bbbk (\Bbbk\langle 1 \rangle \oplus \Bbbk\langle 3 \rangle) \to \mathcal{E}^{(2)}(\lambda) \otimes_\Bbbk (\Bbbk\langle -3 \rangle \oplus \Bbbk\langle -1 \rangle)$$

and there are no negative homological degree endomorphisms of $\mathcal{E}^{(2)}$ (since it is a sheaf) so $T(\lambda)^2 = 0$.

To prove relation (ii) note that

$$\mathcal{E}(\lambda+2) * \mathcal{E}(\lambda) * \mathcal{E}(\lambda-2) \cong \mathcal{E}^{(3)}(\lambda) \otimes \left(\Bbbk\langle -3\rangle \oplus \Bbbk\langle -1\rangle^{\oplus 2} \oplus \Bbbk\langle 1\rangle^{\oplus 2} \oplus \Bbbk\langle 3\rangle\right).$$

Since $\mathrm{End}(\mathcal{E}^{(3)}(\lambda)) \cong \Bbbk \cdot \mathrm{I}$ this means

$$\mathrm{Hom}(\mathcal{E}(\lambda+2) * \mathcal{E}(\lambda) * \mathcal{E}(\lambda-2)\langle 3\rangle, \mathcal{E}(\lambda+2) * \mathcal{E}(\lambda) * \mathcal{E}(\lambda-2)\langle -3\rangle) \cong \Bbbk.$$

Thus $(I * T(\lambda-1)) \circ (T(\lambda+1) * I) \circ (I * T(\lambda-1))$ and $(T(\lambda+1) * I) \circ (I * T(\lambda-1)) \circ (T(\lambda+1) * I)$ must be non-zero multiples of each other or one of them is zero. The rest of the argument below follows formally from relations (iii). Note that we will not use relation (ii) in the proof of (iii).

Lemma 5.10. $(T(\lambda+1) * I) \circ (I * T(\lambda-1)) \neq 0$.

Proof. We have (using shorthand notation)

$$\begin{aligned} X(\lambda-2) \circ T(\lambda+1) \circ T(\lambda-1) &= T(\lambda+1) \circ X(\lambda-2) \circ T(\lambda-1) \\ &= T(\lambda+1) \circ (T(\lambda-1) \circ X(\lambda) - I) \\ &= -T(\lambda+1) + T(\lambda+1) \circ T(\lambda-1) \circ X(\lambda) \end{aligned}$$

where we use that $X(\lambda-1) = I * I * X(\lambda-1)$ and $T(\lambda+1) = T(\lambda+1) * I$ commute to get the first equality. So if $T(\lambda+1) \circ T(\lambda-1) = 0$ then $T(\lambda+1) = 0$ (contradiction). □

Similar to above we obtain

$$X(\lambda-2) \circ T(\lambda-1) \circ T(\lambda+1) \circ T(\lambda-1) = T(\lambda-1) \circ T(\lambda+1) \circ T(\lambda-1) \circ X(\lambda+2) - T(\lambda+1) \circ T(\lambda-1). \tag{10}$$

Notice that this means $T(\lambda-1) \circ T(\lambda+1) \circ T(\lambda-1) \neq 0$ because $T(\lambda+1) \circ T(\lambda-1) \neq 0$.

Again by similar manipulations we obtain,

$$X(\lambda-2) \circ T(\lambda+1) \circ T(\lambda-1) \circ T(\lambda+1) = T(\lambda+1) \circ T(\lambda-1) \circ T(\lambda+1) * X(\lambda+2) - T(\lambda+1) \circ T(\lambda-1). \tag{11}$$

Thus $T(\lambda+1) \circ T(\lambda-1) \circ T(\lambda+1) \neq 0$. So $T(\lambda-1) \circ T(\lambda+1) \circ T(\lambda-1) = \mu(T(\lambda+1) \circ T(\lambda-1) \circ T(\lambda+1))$ for some $\mu \in \Bbbk^\times$. Combining (10) and (11) we obtain that $\mu(T(\lambda+1) \circ T(\lambda-1)) = T(\lambda+1) \circ T(\lambda-1) \neq 0$ so $\mu = 1$ and we are done.

5.3. **Some final isomorphisms.** Having proved Theorem 5.1 we need to finish the proof of the main Theorem 2.7 by checking that certain maps induce isomorphisms.

The first of these is that

$$\oplus_{i=0}^r (X(\lambda+r)^i * I) \circ \iota\langle -2i\rangle : \mathcal{E}^{(r+1)}(\lambda) \otimes_\Bbbk H^\star(\mathbb{P}^r) \to \mathcal{E}(\lambda+r) \circ \mathcal{E}^{(r)}(\lambda-1)$$

and

$$\oplus_{i=0}^r \pi\langle 2i\rangle * (X(\lambda+r)^i \circ I) : \mathcal{E}(\lambda+r) \circ \mathcal{E}^{(r)}(\lambda-1) \to \mathcal{E}^{(r+1)}(\lambda) \otimes_\Bbbk H^\star(\mathbb{P}^r)$$

induce isomorphisms. Fortunately, the first isomorphism follows immediately from Lemma 4.5 because

$$X(\lambda+r) * I = (a(\lambda+r)\theta(\lambda+r+1), b(\lambda+r)\theta(\lambda+r-1), 0)$$

induces the same map (up to non-zero multiple) on cohomology as $\Theta(\lambda-1+r) = (0, \theta(\lambda-1+r), 0)$ (here we used that $b(\lambda+r) = -a(\lambda+r-2) \neq 0$). The proof of the second isomorphism is the same.

The second thing we need to check is that for $\lambda \leq 0$

$$\sigma + \sum_{j=0}^{-\lambda-1} (I * X(\lambda+1)^j \langle -2j\rangle) \circ \eta : \mathcal{E}(\lambda-1) * \mathcal{F}(\lambda-1) \oplus \mathcal{O}_\Delta \otimes_\Bbbk H^\star(\mathbb{P}^{-\lambda-1}) \xrightarrow{\sim} \mathcal{F}(\lambda+1) * \mathcal{E}(\lambda+1)$$

induces an isomorphism (and similarly if $\lambda \geq 0$). Now

$$\sum_{j=0}^{-\lambda-1} (I * X(\lambda+1)^j \langle -2j\rangle) \circ \eta : \mathcal{O}_\Delta \otimes_\Bbbk H^\star(\mathbb{P}^{-\lambda-1}) \to \mathcal{F}(\lambda+1) * \mathcal{E}(\lambda+1)$$

induces an isomorphism $\mathcal{O}_\Delta \otimes_{\Bbbk} H^\star(\mathbb{P}^{-\lambda-1}) \to \mathcal{P}$ by Lemma 4.8. More precisely, Lemma 4.8 says that $\sum_{j=0}^{-\lambda-1} (\Theta(\lambda+2)^j \langle -2j \rangle) \circ \eta$ induces such an isomorphism where $\Theta(\lambda+2) = (0, \theta(\lambda+2), 0)$. But we know that the map $\mathcal{P}\langle -1 \rangle \xrightarrow{(0,0,\theta(\lambda)^j)} \mathcal{P}\langle -1+2j \rangle$ is equal to

$$\mathcal{P} * (\mathcal{O}_{\Delta(\lambda)}\langle -1 \rangle \xrightarrow{\theta(\lambda)^j} \mathcal{O}_{\Delta(\lambda)}\langle -1+2j \rangle)$$

and so it induces zero at the level of cohomology. This means that

$$I * X(\lambda+1)^j = (0, a(\lambda+1)\theta(\lambda+2), b(\lambda+1)\theta(\lambda))^j$$

and $\Theta(\lambda+2)^j$ must induce the same map at the level of cohomology (up to a non-zero multiple).

It remains to show that

$$\sigma : \mathcal{E}(\lambda-1) * \mathcal{F}(\lambda-1) \to \mathcal{F}(\lambda+1) * \mathcal{E}(\lambda+1) \cong \mathcal{E}(\lambda-1) * \mathcal{F}(\lambda-1) \oplus \mathcal{P}$$

induces the zero map $\mathcal{E}(\lambda-1) * \mathcal{F}(\lambda-1) \to \mathcal{P}$ and an isomorphism on $\mathcal{E}(\lambda-1) * \mathcal{F}(\lambda-1)$. The first claim follows from Lemma 4.7.

Too see the second claim it suffices to show that $\sigma \neq 0$ because of Lemma 5.11. To see this we look at the composition

$$\begin{aligned}
(I * X(\lambda+1)) \circ \sigma &= (I * X(\lambda+1)) \circ (I * I * \epsilon) \circ (I * T(\lambda) * I) \circ (\eta * I * I) \\
&= (I * I * \epsilon) \circ (I * X(\lambda+1) * I * I) \circ (I * T(\lambda) * I) \circ (\eta * I * I) \\
&= (I * I * \epsilon) \circ (I + (I * T(\lambda) * I) \circ (I * I * X(\lambda-1) * I)) \circ (\eta * I * I) \\
&= (I * I * \epsilon) \circ (\eta * I * I) + \sigma \circ (X(\lambda-1) * I)
\end{aligned}$$

where we used nil affine Hecke relation (iii) to get the second last line. Now if σ induces zero then this means that

$$(I * I * \epsilon) \circ (\eta * I * I) = \eta \circ \epsilon : \mathcal{E}(\lambda-1) * \mathcal{F}(\lambda-1) \to \mathcal{F}(\lambda+1) \circ \mathcal{E}(\lambda+1)\langle 2 \rangle$$

induces zero. But this map is the composition

$$\mathcal{E}(\lambda-1) * \mathcal{F}(\lambda-1) \xrightarrow{\epsilon} \mathcal{O}_\Delta\langle -\lambda+1 \rangle \xrightarrow{\eta} \mathcal{F}(\lambda+1) * \mathcal{E}(\lambda+1)\langle 2 \rangle$$

where the second map is an inclusion of $\mathcal{O}_\Delta\langle -\lambda+1 \rangle$ into the bottom of $\mathcal{O}_\Delta \otimes_{\Bbbk} H^\star(\mathbb{P}^{-l-1})\langle 2 \rangle$. Thus $\eta \circ \epsilon \neq 0$ and thus $\sigma \neq 0$.

Lemma 5.11. *If* $\lambda \leq 0$ *then*

$$\mathrm{Hom}(\mathcal{E}(\lambda-1) * \mathcal{F}(\lambda-1), \mathcal{E}(\lambda-1) * \mathcal{F}(\lambda-1)[i]\{j\}) = \begin{cases} 0 \text{ if } i < 0 \text{ or } i = 0 \neq j \\ \Bbbk \cdot \mathrm{I} \text{ if } i = 0 = j \end{cases}$$

while if $\lambda \geq 0$ *then*

$$\mathrm{Hom}(\mathcal{F}(\lambda+1) * \mathcal{E}(\lambda+1), \mathcal{F}(\lambda+1) * \mathcal{E}(\lambda+1)[i]\{j\}) = \begin{cases} 0 \text{ if } i < 0 \text{ or } i = 0 \neq j \\ \Bbbk \cdot \mathrm{I} \text{ if } i = 0 = j. \end{cases}$$

Proof. Suppose $\lambda \leq 0$. Then

$$\begin{aligned}
&\mathrm{Hom}(\mathcal{E}(\lambda-1) * \mathcal{F}(\lambda-1), \mathcal{E}(\lambda-1) * \mathcal{F}(\lambda-1)[i]\{j\}) \\
&\cong \mathrm{Hom}(\mathcal{F}(\lambda-1), \mathcal{F}(\lambda-1) * \mathcal{E}(\lambda-1) * \mathcal{F}(\lambda-1)[\lambda-1+i]\{-\lambda+1+j\}) \\
&\cong \mathrm{Hom}(\mathcal{F}(\lambda-1)[-\lambda+1-i]\{\lambda-1-j\}, \mathcal{E}(\lambda-3) * \mathcal{F}(\lambda-3) * \mathcal{F}(\lambda-1) \oplus \mathcal{F}(\lambda-1) \otimes_{\Bbbk} H^\star(\mathbb{P}^{-\lambda+1})).
\end{aligned}$$

Now $\mathrm{Hom}(\mathcal{F}(\lambda-1)[-\lambda+1-i]\{\lambda-1-j\}, \mathcal{F}(\lambda-1) \otimes_{\Bbbk} H^\star(\mathbb{P}^{-\lambda+1}))$ is zero if $i<0$ or $i=0\neq j$ and $\Bbbk$ if $i=0=j$. Also,

$$\begin{aligned}&\mathrm{Hom}(\mathcal{F}(\lambda-1)[-\lambda+1-i]\{\lambda-1-j\}, \mathcal{E}(\lambda-3) * \mathcal{F}(\lambda-3) * \mathcal{F}(\lambda-1))\\ &\cong \mathrm{Hom}(\mathcal{F}(\lambda-3)[-\lambda+3]\{\lambda-3\} * \mathcal{F}(\lambda-1)[-\lambda+1-i]\{\lambda-1-j\}, \mathcal{F}(\lambda-3) * \mathcal{F}(\lambda-1))\\ &\cong \mathrm{Hom}(\mathcal{F}^{(2)}(\lambda-2) \otimes_{\Bbbk} H^\star(\mathbb{P}^1), \mathcal{F}^{(2)}(\lambda-2) \otimes_{\Bbbk} H^\star(\mathbb{P}^1)[2\lambda-4+i]\{-2\lambda+4+j\})\end{aligned}$$

is zero since $2\lambda-4+i<-2$. The result follows.

The case $\lambda \geq 0$ is proved similarly. □

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